Matching in Networks with Bilateral Contracts: Corrigendum*

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Abstract

Hatfield and Kominers (2012) introduced a model of matching in networks with bilateral contracts and showed there that stable outcomes exist in supply chains when firms' preferences over contracts are *fully substitutable*. Hatfield and Kominers (2012) also asserted that in their setting, full substitutability is equivalent to the assumption that all indirect utility representations of each firm's preferences are quasisubmodular; we show here that this equivalence result does not hold in general. We show instead that full substitutability is equivalent to *weak quasisubmodularity* of all indirect utility representations.

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Hatfield and Kominers (2012) introduced a model of matching in networks with bilateral contracts that extends the Ostrovsky (2008) supply chain matching framework. Hatfield and Kominers (2012) showed that stable outcomes exist in supply chains whenever firms' preferences over contracts satisfy a *full substitutability* condition that generalizes the Kelso and Crawford (1982) gross substitutability condition (see also Ostrovsky (2008)).

Hatfield and Kominers (2012) also offered a characterization of full substitutability in their setting, claiming that full substitutability is equivalent to the assumption that all indirect utility representations of each firm's preferences are quasisubmodular. We show here that the claimed equivalence result does not hold in general. We repair the result, showing instead that full substitutability is equivalent to a condition on indirect utility representations that we call *weak quasisubmodularity*.

1 Model

We let F be a finite set of firms, and let X be a finite set of contracts, with each contract x associated to a seller $x_S \in F$ and a buyer $x_B \in F$. We denote by

$$Y_{\to f} \equiv \{y \in Y : y_B = f\} \quad \text{and} \quad Y_{f\to} \equiv \{y \in Y : y_S = f\}$$

the sets of contracts in Y in which f acts respectively as the buyer and the seller. We let $Y_f \equiv Y_{\to f} \cup Y_{f\to}$ denote the set of contracts in Y associated with firm f.[1] We then let $\succ_f$ be a strict preference on $\wp(X_f)$,[2] and let $C^f : \wp(X_f) \to \wp(X_f)$ denote the induced choice function[3]

$$C^f(Y) \equiv \max_{\succ_f} \wp(Y_f).$$

[1] In the original Hatfield and Kominers (2012) paper, the object Y_f was denoted $Y|_f$.

[2] We use $\wp(Y)$ to represent the power set of Y.

[3] We use the notation $\max_{\succ_f}$ to denote that the maximization is taken with respect to $\succ_f$.

We also work with the "buyer component" of f's choice function, which indicates what f buys when it has access to the contracts in Y as a buyer and Z as a seller:

$$C_B^f(Y|Z) \equiv \{x \in C^f(Y_{\to f} \cup Z_{f\to}) : x_B = f\}.$$

Analogously, we define

$$C_S^f(Z|Y) \equiv C^f(Y_{\to f} \cup Z_{f\to}) : x_S = f\}.$$

We also define the associated **rejected sets** of contracts as

$$R_B^f(Y|Z) \equiv Y \smallsetminus C_B^f(Y|Z),$$
$$R_S^f(Z|Y) \equiv Z \smallsetminus C_S^f(Z|Y).$$

Definition 1. The preferences of $f \in F$ are **same-side substitutable** if for all $Y' \subseteq Y \subseteq X_{\to f}$ and $Z' \subseteq Z \subseteq X_{f\to}$, we have that

1. $R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$ and that
2. $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$.

Definition 2. The preferences of $f \in F$ are **cross-side complementary** if for all $Y' \subseteq Y \subseteq X_{\to f}$ and $Z' \subseteq Z \subseteq X_{f\to}$, we have that

1. $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$ and that
2. $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$.

If firm f's preferences are both same-side substitutable and cross-side complementary, then we say that they are **fully substitutable** (Hatfield and Kominers, 2012; Hatfield et al., 2013, Forthcoming).

Hatfield and Kominers (2012) related full substitutability to the indirect utility function without the assumption that there is a numéraire under which preferences are quasilinear: First, Hatfield and Kominers (2012) defined the **offer vector** associated to a set $Y \subseteq X_f$ of contracts by

$$\mathbf{q}^f_x(Y) = \begin{cases} 0 & \text{if } x \in Y \\ 1 & \text{if } x_S = f \text{ and } x \notin Y \\ -1 & \text{if } x_B = f \text{ and } x \notin Y. \end{cases}$$

An indirect utility function u **represents** the preference relation $\succ_f$ (or, more generally, **represents** the choice function C^f) if

$$u(\mathbf{q}^f(Y)) > u(\mathbf{q}^f(Y')) \iff C^f(Y) \succ_f C^f(Y').$$

Hatfield and Kominers (2012) asserted that preferences are fully substitutable if and only if every associated indirect utility representation is quasisubmodular.

Definition 3 (Hatfield and Kominers, 2012)**.** An indirect utility function u is **quasisubmodular** if for all $\mathbf{r} \leq \mathbf{r}'$ and $\mathbf{s} \geq 0$, we have that

$$\begin{aligned} u(\mathbf{r}' + \mathbf{s}) > u(\mathbf{r}') &\Rightarrow u(\mathbf{r} + \mathbf{s}) > u(\mathbf{r}) \\ u(\mathbf{r} + \mathbf{s}) < u(\mathbf{r}) &\Rightarrow u(\mathbf{r}' + \mathbf{s}) < u(\mathbf{r}'). \end{aligned} \tag{1}$$

Claim 1 (stated as Theorem 1 by Hatfield and Kominers (2012))**.** *The preferences of $f \in F$ are fully substitutable if and only if every indirect utility function representing those preferences is quasisubmodular.*

Unfortunately, however, Claim 1 is not quite correct as stated. Here, we provide a counterexample to Claim 1. Then, we explain how to repair the result by weakening the quasisubmodularity condition slightly.

2 Counterexample to Claim 1

Suppose that $X = \{x, \hat{x}, z\}$ and that $f = x_S = \hat{x}_S = z_B$; that is, there are three contracts x, $\hat{x}$, and z and f is the seller for x and $\hat{x}$ and the buyer for z. Let the preferences of f be given by

$$\{\hat{x}, x, z\} \succ_f \{\hat{x}, x\} \succ_f \{\hat{x}, z\} \succ_f \{x, z\} \succ_f \{z\} \succ_f \{\hat{x}\} \succ_f \{x\} \succ_f \varnothing.$$

Note that $C^f(Y) = Y$ for all $Y \subseteq X$; hence, C^f is trivially fully substitutable. However, there exists an indirect utility function representing C^f that is not quasisubmodular.

Indeed, consider the indirect utility function u given by

$u(\mathbf{r}^f(Y))$	$Y_S = \varnothing$	$Y_S = \{x\}$	$Y_S = \{\hat{x}\}$	$Y_S = \{x, \hat{x}\}$
$Y_B = \varnothing$	0	1	2	6
$Y_B = \{z\}$	3	4	5	7

.

Note that when x and $\hat{x}$ are available to f, z becoming available while x becomes unavailable harms f; by contrast, when only x is available to f, z becoming available while x becomes unavailable makes f better off.

We claim that u is not quasisubmodular. Let $\mathbf{r} = (0, 0, -1)$;[4] this corresponds to f having x and $\hat{x}$ available but not z. Let $\mathbf{r}' = (0, 1, -1)$; this corresponds to f having only x available. Finally, let $\mathbf{s} = (1, 0, 1)$; this corresponds to making x unavailable and making z available. Note that $\mathbf{r} \leq \mathbf{r}'$ and that $\mathbf{s} \geq 0$. However,

$$u(\mathbf{r}) = u(\mathbf{q}^f(\{x, \hat{x}\})) = 6 > 5 = u(\mathbf{q}^f(\{\hat{x}, z\})) = u(\mathbf{r} + \mathbf{s})$$
$$u(\mathbf{r}') = u(\mathbf{q}^f(\{x\})) = 1 < 3 = u(\mathbf{q}^f(\{z\})) = u(\mathbf{r}' + \mathbf{s});$$

together, these inequalities violate the second condition of Definition 3.

In their argument for Claim 1, Hatfield and Kominers (2012) asserted that it suffices

[4] We use the convention that the components of the offer vector are given in the order x, $\hat{x}$, z.

to verify (1) for vectors $\mathbf{s}$ that have only one nonzero component. This assertion is not correct—indeed, for the indirect utility function u of our example, (1) holds whenever $\mathbf{r} \leq \mathbf{r}'$ and $\mathbf{s}$ has only one nonzero component, but u is not quasisubmodular.

The key issue is that $\mathbf{s}$ can be positive while both making opportunities (z) available on the buy-side and making opportunities (x) unavailable on the sell-side. When we add just buy-side options to an opportunity set, full substitutability requires that, if those new options are valuable to the agent, they are still valuable when his set of buy-side options shrinks and his set of sell-side options expands; in this case, the inequalities in (1) hold naturally. But when we simultaneously change buy-side and sell-side opportunities, while full substitutability still constrains how the chosen set of contracts can change, it does not tell us whether the changes in the opportunities available to the agent are valuable.

3 Corrected Result

3.1 Set-Theoretic Formulation

To repair the Hatfield and Kominers (2012) equivalence result, we reformulate quasisubmodularity to consider a specific ordering over sets of contracts. For two sets of contracts $Y, Y' \subseteq X$, we say that Y is (weakly) *below* Y' for f if both $Y_{f\to} \subseteq Y'_{f\to}$ and $Y_{\to f} \supseteq Y'_{\to f}$; we denote this relation by $Y \sqsubseteq Y'$.[5,6] We introduce indirect utility functions $v : \wp(X_f) \to \mathbb{R}$ defined over sets of contracts and say that an indirect utility function v **represents** the choice function C^f if

$$v(Y) > v(Y') \Leftrightarrow C^f(Y) \succ_f C^f(Y').$$

Theorem 1. *The preferences of $f \in F$ are fully substitutable if and only if for every indirect*

[5] In settings in which contracts specify prices, the analogue of the relation $Y \sqsubseteq Y'$ is that prices are lower in Y than in Y' (see Appendix A of Fleiner et al. (Forthcoming)).

[6] Note that $Z \sqsupseteq \varnothing$ if and only if $Z \subseteq X_{f\to}$ and, similarly, $Z \sqsubseteq \varnothing$ if and only if $Z \subseteq X_{\to f}$.

utility function v *and all sets* Y, Y', Z *of contracts with* $Y \sqsubseteq Y'$*:*

$$\begin{aligned} &\textit{If } Z \sqsupseteq \varnothing \textit{ then } v(Y' \cup Z) > v(Y') \Rightarrow v(Y \cup Z) > v(Y). \\ &\textit{If } Z \sqsubseteq \varnothing \textit{ then } v(Y \cup Z) > v(Y) \Rightarrow v(Y' \cup Z) > v(Y'). \end{aligned} \tag{2}$$

The key difference between Claim 1 and Theorem 1 is that Condition (2) places restrictions on how changes in either buy- or sell-side opportunities can affect indirect utility, but places no restriction on how simultaneous changes in the available buy- and sell-side opportunities affect indirect utility.

Proof of Theorem 1. First, suppose the preferences of f are fully substitutable. We show that (2) must hold for all sets Y, Y', Z of contracts with $Y \sqsubseteq Y'$. Let $Y \sqsubseteq Y'$ be sets of contracts. Suppose that $Z \sqsupseteq \varnothing$—note that this implies that $Z \subseteq X_{f\to}$. If $v(Y' \cup Z) > v(Y')$, then we have that $C^f(Y' \cup Z) \succ_f C^f(Y')$, and so by revealed preference there exists $z \in Z \smallsetminus Y'$ with $z \in C^f(Y' \cup Z)$.[7] Hence, as the preferences of f are fully substitutable, $Y_{f\to} \subseteq Y'_{f\to}$, $Y_{\to f} \supseteq Y'_{\to f}$, and $Z \subseteq X_{f\to}$, we have that $z \in C^f(Y \cup Z)$. Thus, since $z \notin Y$ (as $z \notin Y'$ and $Y_{f\to} \subseteq Y'_{f\to}$), we have that $C^f(Y \cup Z) \succ_f C^f(Y)$. By revealed preference, it follows that $v(Y \cup Z) > v(Y)$.

An analogous argument shows that $v(Y \cup Z) > v(Y) \Rightarrow v(Y' \cup Z) > v(Y')$ if $Z \sqsubseteq \varnothing$. Hence, (2) must hold for all sets of contracts Y, Y', Z with $Y \sqsubseteq Y'$.

Second, suppose the preferences of f are not fully substitutable. We show that (2) must fail for some sets of contracts Y, Y', Z with $Y \sqsubseteq Y'$. Suppose that the preferences of f violate the first condition of same-side substitutability. Then, there must exist contracts $x, z \in X$ and a set $Y' \subseteq X$ of contracts such that $x_B = z_B = f$ and

$$z \notin C^f(Y' \cup \{z\}) \text{ but } z \in C^f(\{x\} \cup Y' \cup \{z\}).$$

[7] In particular, because $C^f(Y' \cup Z) \succ_f C^f(Y')$ there must exist some contract z that is in $C^f(Y' \cup Z)$ but is not in Y', as otherwise $C^f(Y' \cup Z)$ would have been chosen from Y' and thus $C^f(Y' \cup Z)$ would be the same as $C^f(Y')$.

Taking $Z = \{z\}$ and $Y = Y' \cup \{x\}$, we have by revealed preference that $C^f(Y' \cup Z) = C^f(Y')$ and that $C^f(Y \cup Z) \succ_f C^f(Y)$. Thus, by revealed preference, we have that $v(Y' \cup Z) = v(Y')$ while $v(Y \cup Z) > v(Y)$. As $Z \sqsubseteq \varnothing$ and $Y \sqsubseteq Y'$, this situation constitutes a violation of (2).

Next, suppose that the preferences of f violate the first condition of cross-side complementarity. Then, there must exist contracts $x, z \in X$ and a set $Y' \subseteq X$ of contracts such that $x_B = z_S = f$ and

$$z \in C^f(Y' \cup \{z\}) \text{ but } z \notin C^f(\{x\} \cup Y' \cup \{z\}).$$

Taking $Z = \{z\}$ and $Y = W \cup \{x\}$, we have by revealed preference that $C^f(Y \cup Z) = C^f(Y)$ and that $C^f(Y' \cup Z) \succ_f C^f(Y')$. Thus, by revealed preference, we have that $v(Y \cup Z) = v(Y)$ while $v(Y' \cup Z) > v(Y')$. As $Z \sqsupseteq \varnothing$ and $Y \sqsubseteq Y'$, this situation constitutes a violation of (2).

Analogous arguments show (2) that must fail for some sets Y, Y', Z of contracts with $Y \sqsubseteq Y'$ if the preferences of f violate the second condition of same-side substitutability or the second condition of cross-side complementarity. Hence, (2) that must fail for some sets Y, Y', Z of contracts with $Y \sqsubseteq Y'$ if the preferences of f are not fully substitutable. □

Note that our proof of Theorem 1 also shows that full substitutability is equivalent to the existence of an indirect utility representation that satisfies (2).

3.2 Lattice-Theoretic Formulation

As Gul and Stacchetti (1999) and Hatfield et al. (2013) have shown, lattice-theoretic formulations of the gross substitutability and full substitutability conditions are useful in studying the set of equilibrium price vectors in transferable utility economies (see also Hatfield et al. (Forthcoming)). We therefore offer a lattice-theoretic formulation of Theorem 1.

First, we recall the general definition of quasisubmodularity in terms of lattices.

Definition 4 (Milgrom and Shannon, 1994)**.** An indirect utility function v is **quasisub-**

modular with respect to $\sqsubseteq$ if for all Y and Y' we have that:

$$\text{If } v(Y' \vee Y) > v(Y') \text{ then } v(Y) > v(Y \wedge Y'). \tag{3}$$

$$\text{If } v(Y \wedge Y') > v(Y) \text{ then } v(Y') > v(Y' \vee Y). \tag{4}$$

Here, $\wedge$ and $\vee$ are the meet and join with respect to $\sqsubseteq$.

Quasisubmodularity weakens submodularity (in the sense of Topkis (1998)) by placing a condition on the signs of differences in values instead of on the magnitudes of differences in values. Our weak quasisubmodularity condition weakens quasisubmoularity by only requiring that (3) and (4) hold if $Y_{f\to} \subseteq Y'_{f\to}$ or $Y_{\to f} \supseteq Y'_{\to f}$.

Definition 5. An indirect utility function v is **weakly quasisubmodular with respect to** $\sqsubseteq$ if (3) and (4) holds for all Y and Y' such that $Y_{f\to} \subseteq Y'_{f\to}$ or $Y_{\to f} \supseteq Y'_{\to f}$.

It turns out that weak quasisubmodularity provides a lattice-theoretic formulation of the condition on indirect utility functions introduced in Theorem 1.

Proposition 1. *An indirect utility function $v : \wp(X_f) \to \mathbb{R}$ is weakly quasisubmodular if and only if (2) holds for all sets Y, Y', Z of contracts with $Y \sqsubseteq Y'$.*

Proof. First, suppose that v is a weakly quasisubmodular indirect utility function. We show that (2) must hold for all sets of contracts Y, Y', Z with $Y \sqsubseteq Y'$. Suppose that $Z \sqsupseteq \varnothing$—i.e., $Z \subseteq X_{f\to}$—and that $v(Y' \cup Z) > v(Y')$. As $Y \sqsubseteq Y'$, we have that

$$Y' \vee (Y \cup Z) = Y' \cup Z \quad \text{and that} \quad Y' \wedge (Y \cup Z) \supseteq Y.$$

Hence, since $v(Y' \cup Z) > v(Y')$ by assumption in (2), we have that $v(Y' \vee (Y \cup Z)) > v(Y')$. Weak quasisubmodularity of v then implies that $v(Y \cup Z) > v(Y' \wedge (Y \cup Z))$. As v is an indirect utility function and $Y' \wedge (Y \cup Z) \supseteq Y$, we have that $v(Y' \wedge (Y \cup Z)) \geq v(Y)$ and so

$$v(Y \cup Z) > v(Y' \wedge Y'') \geq v(Y)$$

as desired.

An analogous argument shows that $v(Y \cup Z) > v(Y)$ implies that $v(Y' \cup Z) > v(Y')$ for $Z \sqsubseteq \varnothing$. Hence, (2) holds for all sets of contracts Y, Y', Z with $Y \sqsubseteq Y'$.

Second, suppose that v is an indirect utility function that satisfies (2). We show that v must be weakly quasisubmodular. Let Y and Y' be sets of contracts with $Y_{f\to} \subseteq Y'_{f\to}$. In this case, we have that $[Y \wedge Y']_{f\to} = Y_{f\to}$ and that $[Y \wedge Y']_{\to f} \supseteq Y_{\to f}$. Letting $Z = [Y' \setminus Y]_{\to f}$, we have that

$$Y' = (Y \vee Y') \cup Z \supseteq Y \vee Y'$$

by construction. As v is an indirect utility function, we therefore must have that $v(Y') \geq v(Y' \vee Y)$ and so (3) must hold vacuously as the antecedent is never satisfied.

Furthermore, we have that $Y' \sqsubseteq Y \vee Y'$ and that $Z \sqsubseteq \varnothing$. If $v(Y \wedge Y') > v(W)$, then we have that

$$v(Y \cup Z) = v(Y \wedge Y') > v(Y).$$

Equation (2) then implies that $v((Y \vee Y') \cup Z) > v(Y \vee Y')$ and so

$$v(Y') = v((Y \vee Y') \cup Z \cup Z) > v(Y \vee Y').$$

Hence, we have that

$$\text{If } v(Y \wedge Y') > v(Y) \text{ then } v(Y') > v(Y' \vee Y)$$

as well, and so both (3) and (4) hold.

An analogous argument shows that (3) and (4) must hold if $Y_{\to f} \supseteq Y'_{\to f}$. Hence, v must be weakly quasisubmodular. □

Theorem 1 and Proposition 1 together imply that full substitutability is equivalent to the weak quasisubmodularity of every indirect utility representation.

Corollary 1. *The preferences of* $f \in F$ *are fully substitutable if and only if every indirect utility function representing those preferences is weakly quasisubmodular with respect to* $\sqsubseteq$.

Hatfield and Kominers (2017) and Hatfield et al. (Forthcoming) have proven similar characterizations of substitutability in different contexts. Specifically, Hatfield and Kominers (2017) and Hatfield et al. (Forthcoming) have shown that (full) substitutability is equivalent to the existence of a submodular indirect utility representation in many-to-many matching without transfers and in trading networks with quasilinear preferences, respectively. Submodularity obtains in the Hatfield and Kominers (2017) setting because each agent is only a buyer or only a seller, and so it is impossible to change both an agent's buy-side and his sell-side opportunities. Meanwhile, submodularity obtains in the Hatfield et al. (Forthcoming) setting as a consequence of quasilinearity.

In a recent paper, Fleiner et al. (Forthcoming) have developed a version of Corollary 1 for a setting with continuous prices and potentially nonquasilinear preferences (see Theorem A.1 of Fleiner et al. (Forthcoming)). Fleiner et al. (Forthcoming) use their equivalence result in the proof of their main existence result; this application is similar to the use of the equivalence between full substitutability and submodularity of the indirect utility function in the work of Hatfield et al. (2013, Forthcoming).

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