Abstract

Hatfield and Kominers (2012) introduced a model of matching in networks with bilateral contracts and showed there that stable outcomes exist in supply chains when firms’ preferences over contracts are fully substitutable. Hatfield and Kominers (2012) also asserted that in their setting, full substitutability is equivalent to the assumption that all indirect utility representations of all firms’ preferences are quasisubmodular; however we show here that, for subtle reasons, this equivalence result does not hold in general. We show instead that full substitutability is equivalent to weak quasisubmodularity of all indirect utility representations.

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Hatfield and Kominers (2012) introduced a model of matching in networks with bilateral contracts that extends the Ostrovsky (2008) supply chain matching framework. Hatfield and Kominers (2012) showed that stable outcomes exist in supply chains whenever firms’ preferences over contracts satisfy a full substitutability condition that generalizes the Kelso and Crawford (1982) gross substitutability condition (see also Ostrovsky (2008); Hatfield and Kominers (2012); Hatfield et al. (Forthcoming)).

Hatfield and Kominers (2012) also offered a characterization of full substitutability in their setting, claiming that full substitutability is equivalent to the assumption that all indirect utility representations of firms’ preferences are quasisubmodular. We show here that, for subtle reasons, the claimed equivalence result does not hold in general. We repair the result, showing instead that full substitutability is equivalent to weak quasisubmodularity of all indirect utility representations.

1 Model

We let $F$ be a finite set of firms, and let $X$ be a finite set of contracts, with each contract $x$ associated to a seller $x_S \in F$ and a buyer $x_B \in F$.

We denote by

$$Y^B|_f \equiv \{y \in Y : y_B = f\} \quad \text{and} \quad Y^S|_f \equiv \{y \in Y : y_S = f\}$$

the sets of contracts in $Y$ in which $f$ acts respectively as the buyer and the seller. We let $Y|_f \equiv Y^B|_f \cup Y^S|_f$ denote the set of contracts in $Y$ associated with firm $f$.\textsuperscript{1} We then let $\succ$ be a strict preference on $\wp(X|_f)$, and let $C^f : \wp(X|_f) \to \wp(X|_f)$ denote the induced choice function\textsuperscript{2}

$$C^f(Y) \equiv \max_{\succ} \wp(Y|_f).$$

\textsuperscript{1}We use $\wp(Y)$ to represent the power set of $Y$.

\textsuperscript{2}The notation $\max_{\succ}$ to denote that the maximization is taken with respect to $P^f$.  

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We also work with the “buyer component” of f’s choice function, which indicates what f buys when it has access to the contracts in $Y \subseteq X^B|_f$ as a buyer and $Z \subseteq X^S|_f$ as a seller:

$$C^f_B(Y|Z) \equiv C^f(Y \cup Z) \cap X^B_f.$$

Analogously, we define

$$C^f_S(Z|Y) \equiv C^f(Y \cup Z) \cap X^S_f.$$

We also define the associated rejected sets of contracts as

$$R^f_B(Y|Z) \equiv Y \setminus C^f_B(Y|Z),$$

$$R^f_S(Z|Y) \equiv Z \setminus C^f_S(Z|Y).$$

**Definition 1.** The preferences of $f \in F$ are **same-side substitutable** if for all $Y' \subseteq Y \subseteq X^B|_f$ and $Z' \subseteq Z \subseteq X^S|_f$,

1. $R^f_B(Y'|Z) \subseteq R^f_B(Y|Z)$ and

2. $R^f_S(Z'|Y) \subseteq R^f_S(Z|Y)$.

**Definition 2.** The preferences of $f \in F$ are **cross-side complementary** if for all $Y' \subseteq Y \subseteq X^B|_f$ and $Z' \subseteq Z \subseteq X^S|_f$,

1. $R^f_B(Y|Z) \subseteq R^f_B(Y|Z')$ and

2. $R^f_S(Z|Y) \subseteq R^f_S(Z|Y')$.

If firm f’s preferences are both same-side substitutable and cross-side complementary, then we say that they are **fully substitutable** (Hatfield and Kominers, 2012; Hatfield et al., 2013, Forthcoming).

Hatfield and Kominers (2012) related full substitutability to the indirect utility function without the assumption that there is a numéraire under which preferences are quasilinear:
First, Hatfield and Kominers (2012) defined the offer vector associated to a set \( Y \subseteq X \mid f \) of contracts by

\[
q^f(Y) = \begin{cases} 
0 & \text{if } x \in Y \\
1 & \text{if } x_S = f \text{ and } x \notin Y \\
-1 & \text{if } x_B = f \text{ and } x \notin Y.
\end{cases}
\]

An indirect utility function \( u^f \) represents the preference relation \( \succ \) (or, more generally, represents the choice function \( C^f \)) if

\[
u(q^f(Y)) > u(q^f(Y')) \iff C^f(Y) \succ C^f(Y').
\]

Hatfield and Kominers (2012) asserted that preferences are fully substitutable if and only if every associated indirect utility representation is quasisubmodular.

**Definition 3.** An indirect utility function \( v^f \) is quasisubmodular if for all \( q \leq q' \) and \( s \geq 0 \), we have that

\[
u(q' + s) > u(q') \implies u(q + s) > u(q)
\]

\[
u(q + s) < u(q) \implies u(q' + s) < u(q').
\]

**Claim 1** (stated as Theorem 1 by Hatfield and Kominers (2012)). The preferences of \( f \in F \) are fully substitutable if and only if every indirect utility function representing these preferences is quasisubmodular.

Unfortunately, however, Claim 1 is not quite correct as stated. Here, we provide a counterexample to Claim 1. Then, we explain how to repair the result by weakening the quasisubmodularity condition slightly.

## 2 Counterexample to Claim 1

Suppose that \( X = \{x, \hat{x}, z\} \) and that \( f = \{x, \hat{x}\}_S = \{z\}_B \); that is, there are three contracts \( x, \hat{x}, \) and \( z \) and \( f \) is the seller for \( x \) and \( \hat{x} \) and the buyer for \( z \). Let the preferences of \( f \) be
given by

\[ P^f : \{\hat{x}, x, z\} \succ \{\hat{x}, x\} \succ \{\hat{x}, z\} \succ \{x, z\} \succ \{\hat{x}\} \succ \{x\} \succ \emptyset. \]

Note that \( C^f(Y) = Y \) for all \( Y \subseteq X \); hence, \( C^f \) is trivially fully substitutable. However, there exists an indirect utility function representing \( C^f \) that is not quasisubmodular.

Indeed, consider the indirect utility function \( u \) given by

\[
\begin{array}{c|c|c|c|c}
 u(q^f(Y)) & Y_S = \emptyset & Y_S = \{x\} & Y_S = \{\hat{x}\} & Y_S = \{x, \hat{x}\} \\
\hline
 Y_B = \emptyset & 0 & 1 & 2 & 6 \\
 Y_B = \{z\} & 3 & 4 & 5 & 7 \\
\end{array}
\]

Note that, when \( x \) and \( \hat{x} \) are available to \( f \), \( z \) becoming available while \( x \) becomes unavailable harms \( f \); by contrast, when only \( x \) is available to \( f \), \( z \) becoming available while \( x \) becomes unavailable makes \( f \) better off.

We claim that \( u \) is not quasisubmodular. Let \( q = (0, 0, -1) \); this corresponds to \( f \) having \( x \) and \( \hat{x} \) available but not \( z \). Let \( q' = (0, 1, -1) \); this corresponds to \( f \) having only \( x \) available. Finally, let \( s = (1, 0, 1) \); this corresponds to making \( x \) unavailable and making \( z \) available. Note that \( q \leq q' \) and that \( s \geq 0 \). However,

\[
\begin{align*}
 u(q) &= u(q^f(\{x, \hat{x}\})) = 6 > 5 = u(q^f(\{\hat{x}, z\})) = u(q + s) \\
 u(q') &= u(q^f(\{x\})) = 1 < 3 = u(q^f(\{z\})) = u(q' + s); \\
\end{align*}
\]

together, these inequalities violate the second condition of Definition 3.

In the proof of Claim 1, Hatfield and Kominers (2012) assert that it suffices to verify (1) for vectors \( s \) that have only one nonzero component. This assertion is not correct—for example, (1) holds whenever \( q \leq q' \) and \( s \) has only one nonzero component, but \( u \) is not quasisubmodular.

\footnote{We use the convention that the components of the offer vector are given in the order \( x, \hat{x}, z \).}
The key issue is that $s$ can be positive while simultaneously making opportunities ($z$) available on the buy-side while making opportunities ($x$) unavailable on the sell-side. As we demonstrate below, if we only considered changes $s'$ wholly on the buy-side or wholly on the sell-side, then we would have $u(q + s') > u(q)$ (for changes wholly on the buy-side) or $u(q' + s') < u(q')$ (for changes wholly on the sell-side).

3 Corrected Result

To repair the Hatfield and Kominers (2012) equivalence result, we weaken quasisubmodularity slightly. Instead of allowing $s \geq 0$ to be arbitrary in Definition 3, we require that $s_x = 0$ either for all $x$ in which $f$ acts as a seller or all $x$ in which $f$ acts as a buyer.

Definition 4. An indirect utility function $u$ is weakly quasisubmodular if for all $q \leq q'$ and $s \geq 0$ with $s_X B_f = 0$ or $s_X S_f = 0$, we have (1), that is,

$$u(q' + s) > u(q') \implies u(q + s) > u(q)$$

$$u(q + s) < u(q) \implies u(q' + s) < u(q').$$

We now show that full substitutability is equivalent to the weak quasisubmodularity of every indirect utility representation.

Theorem 1. The preferences of $f \in F$ are fully substitutable if and only if every indirect utility function representing those preferences is weakly quasisubmodular.

Hatfield and Kominers (2017) and Hatfield et al. (Forthcoming) have proved similar characterizations of substitutability in different contexts. Specifically, Hatfield and Kominers (2017) and Hatfield et al. (Forthcoming) have shown that (full) substitutability is equivalent to the existence of a submodular indirect utility representation in many-to-many matching without transfers and in trading networks with quasilinear preferences, respectively. Submodularity obtains in the Hatfield and Kominers (2017) setting because indirect utility
representations are monotone in two-sided matching environments. Meanwhile, submodularity obtains in the Hatfield et al. (Forthcoming) setting as a consequence of quasilinearity.

In a recent paper, Fleiner et al. (2018) have developed a version of Theorem 1 for a setting with continuous prices and potentially nonquasilinear preferences (see Theorem A.1 of Fleiner et al. (2018)). Fleiner et al. (2018) use their equivalence result in the proof of their main existence result; this is similar to the applications of the equivalence between full substitutability and submodularity of the indirect utility function in the work of Hatfield et al. (2013, Forthcoming).

To prove Theorem 1, we use different arguments for the two directions of the equivalence. The “if” direction of Theorem 1 follows from the proof of the “if” direction of Claim 1 given by Hatfield and Kominers (2012); we thus just present the proof of the “only if” direction.

Proof of the “only if” direction of Theorem 1. Let $u$ be any indirect utility representation of $f$’s preferences. We show that $u$ is weakly quasisubmodular.

Suppose that $q \leq q'$ and $s \geq 0$ are such that $s_{X^B|f} = 0$. Let $Y, Y', Z \subseteq X$ be such that $q = q^f(Y)$, $q' = q^f(Y')$, and $s = q^f(Z)$. By construction, we have that $q + s = q^f(Y \cup Z)$ and that $q' + s = q^f(Y' \cup Z)$.

If $u(q' + s) > u(q')$, then we must have that

$$C^f(Y' \cup Z) \succ C^f(Y')$$

because $u$ is an indirect utility representation. By revealed preference, we must have that $C^f(Y' \cup Z) \cap Z \neq \emptyset$. As $q \leq q'$, we must have that $Y \cap X^S|f \subseteq Y'$ and that $Y' \cap X^B|f \subseteq Y$. As $Z_S = f$, full substitutability therefore implies that $C^f(Y \cup Z) \cap Z \neq \emptyset$. By revealed preference, we must have that $C^f(Y \cup Z) \succ C^f(Y)$, and hence that $u(q + s) > u(q)$ because $u$ is an indirect utility representation. Hence, we have proven that $u(q' + s) > u(q')$ implies that $u(q + s) > u(q)$.

Now, we note that $C^f(Y \cup Z) \succeq C^f(Y)$ holds by revealed preference. Because $u$ is
an indirect utility representation, it follows that \( u(q + s) \geq u(q) \). Hence, we have that \( u(q + s) < u(q) \) implies that \( u(q' + s) < u(q') \)—as \( u(q + s) < u(q) \) can never hold.

We have therefore proven that (1) holds if \( s_{X^B|f} = 0 \). An analogous argument shows that (1) holds if \( s_{X^S|f} = 0 \). Hence, \( u \) must be weakly quasisubmodular, as desired.

\[ \square \]

References


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