

Matching in Networks with Bilateral Contracts

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Abstract

We introduce a model in which firms trade goods via bilateral contracts which specify a buyer, a seller, and the terms of the exchange. This setting subsumes (many-to-many) matching with contracts, as well as supply chain matching. When firms' relationships do not exhibit a supply chain structure, stable allocations need not exist. By contrast, in the presence of supply chain structure, a natural substitutability condition characterizes the maximal domain of firm preferences for which stable allocations always exist. Furthermore, the classical lattice structure, rural hospitals theorem, and one-sided strategy-proofness results all generalize to this setting.

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1 Introduction

The theoretical literature on two-sided matching began with the simple one-to-one (marriage) model of Gale and Shapley (1962), in which agents on opposite sides of a market (men and women) seek to match into pairs. The central solution concept in this literature is *stability*, the requirement that for no unmatched pair does each agent prefer the other to their current assigned partner. Gale and Shapley (1962) showed that stable one-to-one matches exist in general, and obtained conditions under which this existence result is preserved even if agents on one side of the market are allowed to match to multiple partners, that is, when the matching is many-to-one (as in college admissions and doctor-hospital matching). Following high-profile applications of matching in labor markets and school choice programs,¹ the foundational work on matching has been extensively generalized.² Recently, Ostrovsky (2008) illustrated that matching markets need not be two-sided—they may instead consist of a market of firms organized into supply chains. Earlier matching models easily embed into the Ostrovsky (2008) supply chain framework: for example, the many-to-one matching market between doctors and hospitals may be thought of as a “one-step supply-chain” in which doctors sell their services to hospitals.

Although the expanding work on matching has eliminated nearly all the theoretical restrictions imposed in the early literature, two assumptions have been maintained throughout, either implicitly or explicitly:

- *acyclicity* – no agent may both buy from and sell to another agent, even through

¹Roth and Sotomayor (1990) provide a survey of the pre-1990 theory of matching. Roth (2008) gives an updated account, as well as references for historical and recent applications of matching. For examples of specific applications, see the work of Roth and Peranson (1999) (National Resident Matching Program) and Abdulkadiroğlu et al. (2005a,b, 2009) (school choice).

²Kelso and Crawford (1982) extended many-to-one matching to a setting in which matches are supplemented by wage negotiations; Hatfield and Milgrom (2005) generalized this framework still further, by allowing agents to negotiate contracts which fully specify both a matching and the conditions of the match.

Meanwhile, a host of work has studied the existence of stable matchings in many-to-many matching settings, two-sided markets in which all agents may match to multiple partners (as in the matching of consultants to firms). Many-to-many matching has been studied, for example, in the work of Sotomayor (1999, 2004), Echenique and Oviedo (2006), and Konishi and Ünver (2006). Recently, Klaus and Walzl (2009) and Hatfield and Kominers (2010) merged this line of research with that of Hatfield and Milgrom (2005), introducing a theory of many-to-many matching with contracts.

intermediaries, and

- *full substitutability* – upon being endowed with an additional item, an agent’s demand for other items is lower, both in the sense of a reduced desire to buy additional items and an increased desire to sell items he currently owns.³

The acyclicity condition is implicit in all studies of two-sided matching, as the “two sides” may be identified as the set of buyers and sellers, and so each agent acts only as a buyer or only as a seller. Furthermore, acyclicity corresponds to the supply chain structure imposed by Ostrovsky (2008).

Full substitutability generalizes a heavily-studied notion of preference (gross) substitutability first introduced by Kelso and Crawford (1982). Substitutability, in turn, generalizes the responsive preference condition introduced by Gale and Shapley (1962). These successively more-general substitutability conditions have been shown to be essential for the existence of stable allocations in a variety of matching contexts.⁴

Moreover, both acyclicity and full substitutability are natural in most previously-studied matching settings, such as the matching of residents to hospitals (Roth and Peranson (1999)), the assignment of students to schools (Abdulkadiroğlu et al. (2005a,b, 2009)), and the supply-chain production of steel (see Ostrovsky (2008)). Acyclicity may not hold, however, in electricity markets, where an individual firm may buy power from a neighboring firm in one region and sell power to that same firm in another region. Full substitutability is unlikely to apply in settings such as the matching of auto-parts suppliers and assemblers,

³Full substitutability is a condition on firms’ preferences familiar from auction theory. Indeed, full substitutability is an ordinal analogue of the conventional notion of substitutes from auction theory (see, for instance, Milgrom (2004)) and we prove more formally that it is equivalent to quasimodularity of the associated indirect utility function.

⁴The sufficiency and necessity of substitutability for the guaranteed existence of stable allocations holds in the settings of many-to-one matching (Roth (1984) proves sufficiency; Hatfield and Kojima (2008) prove necessity), many-to-many matching (Roth (1984) and Echenique and Oviedo (2006) prove sufficiency; necessity follows from the results of Hatfield and Kojima (2008)), many-to-many matching with contracts (Klaus and Walzl (2009) and Hatfield and Kominers (2010) prove sufficiency; Hatfield and Kominers (2010) prove necessity). Substitutable preferences are sufficient for the existence of a stable outcome in the setting of many-to-one matching with contracts (Hatfield and Milgrom (2005)) but are not necessary (Hatfield and Kojima (2008, 2010)).

where different parts are complementary in the production of the final good.⁵

In this paper, we introduce a matching model in which firms trade via bilateral contracts which specify a buyer, a seller, and the terms of the exchange. This model subsumes all classical matching models, and its generality allows us to make two novel theoretical contributions.

First, we show that both acyclicity and full substitutability are necessary for classical matching theory. If either condition is violated, then stable allocations cannot be guaranteed.⁶ Intuitively, if a contracting relationship contains a cycle, and if a firm in the cycle has an outside option which he prefers to one contract in the cycle, then both the outside option and the complete trading cycle are unstable; the necessity of acyclicity follows. The necessity of full substitutability is more technical to illustrate, but follows closely upon prior results of Hatfield and Kominers (2010).

Second, in the presence of acyclicity and fully substitutable preferences, we fully generalize the key results of classical matching theory. We prove that under these conditions, stable allocations correspond bijectively to fixed points of an isotone operator; Tarski's fixed point theorem then guarantees the existence of a lattice of stable allocations. We also prove a generalization of the classical rural hospitals theorem of Roth (1986) and the strategy-proofness results of Hatfield and Milgrom (2005) and Hatfield and Kojima (2009).⁷ These latter results display a surprising structure which can only be elicited within a framework as general as ours: in particular, we show that the difference between the numbers of buy- and sell-contracts held, rather than the absolute number of contracts held, is invariant across stable allocations for each agent.⁸

⁵In an econometric study of many-to-many matching with transferable utility, Fox (2009) identifies substantial complementarity between inputs into automobile production.

⁶Note that we use a notion of stability which is distinct from *chain stability* introduced by Ostrovsky (2008). Our stability concept is more stringent than chain stability, although these two notions coincide on acyclic contract domains over which firm preferences are fully substitutable (Theorem 7). As we detail in Section 2.3, for domains where these conditions do not hold, chain stability has some unappealing properties.

⁷As in other work in the theory of generalized matching our rural hospitals theorem provides an invariant on agents' net position in contracts. It thus only partially generalizes the original rural hospitals theorem of Roth (1986). However, as we discuss in Footnote 32, this generalization is maximal.

⁸The Roth (1986) rural hospitals theorem and its subsequent generalizations by Hatfield and Milgrom

In light of our necessity results, our work establishes a frontier of matching theory. Without acyclicity and fully substitutable preferences, stable allocations are not guaranteed to exist in general, and hence the results of classical matching theory fail. Up to the failure of these conditions, however, all of the results of classical matching theory hold. Thus, our work in some sense characterizes the set of applications to which classical matching theory applies without imposition of additional structure: Settings where both acyclicity and full substitutability hold—such as labor markets, school choice, and steel production—are directly approachable via classical matching theory; settings where either acyclicity or full substitutability fails—such as electricity and auto-parts markets—are not.

The remainder of this paper is organized as follows. We formalize our model, restrictions on preferences, and solution concepts in Section 2. In Section 3, we prove the sufficiency and necessity of fully substitutable preferences for the existence of stable contract allocations. In Section 4, we discuss the structure of the set of stable allocations, proving our rural hospitals and strategy-proofness results. We conclude in Section 5. All proofs are deferred to the Appendix.

2 Model

There is finite set F of firms, and a finite set X of contracts. Each contract $x \in X$ is associated with both a buyer x_B and a seller x_S ; there may be several contracts with the same buyer and the same seller.⁹

For concreteness, one may suppose each contract $x \in X$ denotes the exchange of a single unit of a good from x_S to x_B .¹⁰ However, contracts need not use a constant unit. For (2005) and Hatfield and Kominers (2010) all showed that, under certain conditions, the *number* of contracts signed by each firm is invariant across stable allocations. The natural conjecture that this exact result would extend to our setting is false, as we demonstrate in Section 4.1. We instead find that the proper invariant for each firm in our framework is the difference between the numbers of buy- and sell-contracts held by that firm. This result implies the previous rural hospitals results because, in a two-sided market, no firm can be both a buyer and a seller.

⁹Note that since X is finite, we may interpret X as being a subset of the set $F \times F \times T$, for some finite set T of possible contract terms. With this notation, a contract $x \in X$ is a 3-tuple: $x = (x_B, x_S, t)$ with $x_B, x_S \in F$ and $t \in T$.

¹⁰In this case, an exchange of 17 units from x_S to x_B would technically occur through 17 different con-

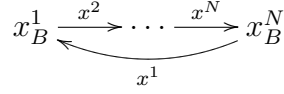


Figure 1: A contract cycle.

example, labor markets might allow both full- and part-time job contracts.¹¹

Let $x_F \equiv \{x_B, x_S\}$ be the set of the firms associated with contract x . For a set of contracts Y , we denote

$$Y_B \equiv \bigcup_{y \in Y} \{y_B\}, \quad Y_S \equiv \bigcup_{y \in Y} \{y_S\}, \quad Y_F \equiv Y_B \cup Y_S.$$

The contract set X is **acyclic** if there does not exist a **cycle**, i.e. a set of contracts

$$\{x^1, \dots, x^N\} \subseteq X$$

such that $x_B^1 = x_S^2, x_B^2 = x_S^3, \dots, x_B^{N-1} = x_S^N, x_B^N = x_S^1$ (as pictured in Figure 1).¹² This condition is equivalent to the condition that there is an ordering \triangleright on F such that for all $x \in X, x_S \triangleleft x_B$; for an acyclic contract set X , if $f \triangleleft f'$, we will say that f is **upstream** of f' and that f' is **downstream** of f . We say that X is **exhaustive** if there is a contract between any two firms, that is, if for all $f \neq f', f, f' \in F$ there exists a contract x such that $x_F = \{f, f'\}$.

Each $f \in F$ has a strict preference relation P^f over sets of contracts involving f . Let $Y|_f \equiv \{y \in Y | f \in y_F\}$ be the set of contracts in Y associated with firm f .

For any $Y \subseteq X$, we first define the **choice set** of f as the set of contracts f chooses from Y . Define

$$C^f(Y) \equiv \max_{P^f} \{Z \subseteq Y | x \in Z \Rightarrow f \in x_F\}.$$

tracts. While in practice the actual sale would not transact in this fashion—a single contractual document would cover the sale of all 17 units—use of primitive contract units lose no generality and will help us interpret the numerical implications of our results. For example, when primitive units are used, our Theorem 8 characterizes the excess stock of goods held by each firm at every stable allocation.

¹¹As we point out in Section 4.1, the practical implications of our results which involve numerical contract counts are unclear if contracts are not denoted in a fixed unit. We present an in-depth discussion of these issues in Hatfield and Kominers (2010).

¹²In our diagrams, an arrow $h_1 \xrightarrow{z} h_2$ between two firms denotes a contract z with seller $z_S = h_1$ and buyer $z_B = h_2$.

It will also be convenient to define the choice function for f as a buyer when f has access to the set of contracts Y as a buyer and the set of contracts Z as a seller. Hence we define

$$C_B^f(Y|Z) \equiv \{x \in C^f(\{y \in Y|y_B = f\} \cup \{z \in Z|z_S = f\}) | x_B = f\}.$$

Analogously, we define

$$C_S^f(Z|Y) \equiv \{x \in C^f(\{y \in Y|y_B = f\} \cup \{z \in Z|z_S = f\}) | x_S = f\}.$$

We also define the **rejected set** of contracts when acting as a buyer or as a seller as

$$R_B^f(Y|Z) \equiv Y - C_B^f(Y|Z),$$

$$R_S^f(Z|Y) \equiv Z - C_S^f(Z|Y).$$

Let $C_B(Y|Z) \equiv \bigcup_{f \in F} C_B^f(Y|Z)$ be the set of contracts chosen from Y by some firm as a buyer, and $C_S(Z|Y) \equiv \bigcup_{f \in F} C_S^f(Z|Y)$ be the set of contracts chosen from Z by some firm as a seller. Let $R_B(Y|Z) \equiv Y - C_B(Z|Y)$ and $R_S(Z|Y) \equiv Z - C_S(Z|Y)$.

An **allocation** is a set of contracts $A \subseteq X$. Preference relations are extended to allocations in a natural way: for two allocations $W, V \subseteq X$, we write $W \succ_f V$ to mean $W|_f \succ_f V|_f$.

2.1 Conditions on Preferences

2.1.1 Full Substitutability

We now proceed to introduce the two conditions on preferences of Ostrovsky (2008). The primary condition on preferences studied in matching theory is *substitutability*. Intuitively, contracts x and y are substitutes for f if they are the same type of contract for f and they are not complements. For example, if $x_B = y_B = f$, and f rejects the contract y from Y as a buyer while having access to Z as a seller, f will not choose y from the larger set $\{x\} \cup Y$ while still having access to Z as a seller. The formal definition of substitutes is given below:

Definition 1. Preferences of $f \in F$ are **same-side substitutable** if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

1. $R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$ and
2. $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$.

Note that this condition is over offer sets; it states that any contract that is rejected from a smaller offer set is also rejected from a larger one.

However, for models where firms can be both buyers and sellers, we must consider how additional offers on one side of the market changes firms' choices on the other side of the market. The key condition here is *cross-side complementarity*. Intuitively, contracts y and z are cross-side complements for f where $y_B = f = z_S$ whenever f chooses y from Y as a buyer when the set Z of contracts is available to f as a seller, f still chooses y from Y when $\{z\} \cup Z$ is available to f as a seller.

Definition 2. Preferences of $f \in F$ are **cross-side complementary** if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

1. $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$ and
2. $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$.

Same-side substitutability and cross-side complementarity are closely linked. For illustration, suppose that each contract delineates the transfer of an object: that is, the transfer of an object from the seller to the buyer. If a firm's preferences are both same-side substitutable and cross-side complementary, then the firm has "substitutable" preferences over objects: The firm is more willing to buy an object if either there are less other objects available to buy (same-side substitutes), or there are more opportunities for the firm to sell objects he already possesses (cross-side complements). Similarly, the firm is more willing to sell an object if either there are fewer other opportunities to sell objects the firm

already possesses (same-side substitutes), or more opportunities to buy an object to replace the one the firm is losing (cross-side complements). In other words, the more objects the firm currently holds, the less willing the firm is to buy/keep new objects. Hence we shall call **fully substitutable** any preference relation that satisfies both same-side substitutes and cross-side complements.

We can characterize the set of preferences which are fully substitutable. We describe the set of contracts that f may choose to sign by the **offer vector** $\vec{q}^f = (\vec{q}_x^f)_{x \in X|_f}$ defined by

$$\vec{q}_x^f(Y) = \begin{cases} 0 & x_B = f \text{ and } x \in Y \\ -1 & x_B = f \text{ and } x \notin Y \\ 0 & x_S = f \text{ and } x \in Y \\ 1 & x_S = f \text{ and } x \notin Y. \end{cases}$$

Intuitively, $\vec{q}_x^f(Y)$ refers to the “object” associated with contract x . A value of 1 is given if the firm currently owns the object but cannot sell it. A value of 0 is given if either the firm does not currently own the object but may choose to buy it, or the firm currently owns the object but may choose to sell it. Finally, a value of -1 is given if the firm does not currently own the object and can not buy it.

Using the above notation, we can represent preferences over offer sets with an indirect utility function u over offer vectors. An indirect utility function u **represents** a preference relation P^f if

$$u(\vec{q}^f(Y)) > u(\vec{q}^f(Y')) \Leftrightarrow C^f(Y) \succ_f C^f(Y') \text{ for all } Y, Y' \subseteq X.$$

Of particular interest are preferences that induce a quasisubmodular indirect utility function.^{13,14}

¹³Recall that $u(\cdot)$ is quasisubmodular if for all $\vec{q} \leq \vec{r}$ and $\vec{s} \geq 0$ we have that

$$\begin{aligned} u(\vec{r} + \vec{s}) - u(\vec{r}) > 0 &\Rightarrow u(\vec{q} + \vec{s}) - u(\vec{q}) > 0, \\ u(\vec{q} + \vec{s}) - u(\vec{q}) < 0 &\Rightarrow u(\vec{r} + \vec{s}) - u(\vec{r}) < 0. \end{aligned}$$

¹⁴The utility function is indirect as it is a function of what contracts are available to the firm, as opposed to what contracts the firm actually chooses.

Theorem 1. *The preferences of $f \in F$ are fully substitutable if and only if every indirect utility function representing these preferences is quasisubmodular.*

Submodularity of the indirect utility function is the key condition in demand theory for preferences to be demand-theory substitutes; see, e.g., Milgrom (2009). However, in the absence of transferable utility it is impossible to quantify the increase in utility from a newly available contract and therefore we can only characterize the utility functions in terms of the ordinal notion of quasisubmodularity.¹⁵

2.1.2 The Laws of Aggregate Supply and Demand

A number of important results in two-sided matching theory rely on the law of aggregate demand, which was first introduced by Hatfield and Milgrom (2005).¹⁶ We generalize this concept to the matching in networks framework with the following definitions:

Definition 3. Preferences of f satisfy the **law of aggregate demand** if for all $Y, Z \subseteq X$ and $Y' \subseteq Y$,

$$\left| C_B^f(Y|Z) \right| - \left| C_B^f(Y'|Z) \right| \geq \left| C_S^f(Z|Y) \right| - \left| C_S^f(Z|Y') \right|,$$

and satisfy the **law of aggregate supply** if for all $Y, Z \subseteq X$ and $Z' \subseteq Z$,

$$\left| C_S^f(Z|Y) \right| - \left| C_S^f(Z'|Y) \right| \geq \left| C_B^f(Y|Z) \right| - \left| C_B^f(Y|Z') \right|.$$

Note that the laws of aggregate demand and supply generalize the analogous definitions from two-sided matching. In two-sided matching, if a firm is a buyer, then that firm does not choose any contracts as a seller, and so the right-hand side of the law of aggregate demand vanishes, and hence the law of aggregate demand reduces to its usual statement,

¹⁵For firms who are either only buyers or only sellers, Hatfield and Kominers (2010) show that if preferences are (same-side) substitutable, then not only is every function that represents these preferences quasisubmodular, but one can always find a submodular function that represents these preferences. This second result relies on a technique introduced by Chambers and Echenique (2009), who show in general that there exists a monotonic transformation for any quasisubmodular utility function that transforms it to a submodular utility function. A similar technique can not be applied to the current setting, as utility is not monotonic in the offer vector.

¹⁶Alkan and Gale (2003) introduced a related condition called “size monotonicity”.

that a firm, upon receiving additional offers, chooses at least as many offers as it did before. However, when a firm can be both a buyer and a seller of contracts, the condition is more subtle. The law of aggregate demand now imposes that when a firm obtains additional offers as a buyer, that firm takes on at least as many new contracts as a buyer as he takes on as a seller. Intuitively, the condition states that when a firm has a new offer where he is a buyer that he accepts, while holding onto his other offers, that firm will then sell at most one new item. Similarly, the law of aggregate supply can be interpreted to say that when the firm receives a new offer to sell that he accepts while holding onto his other offers, he will choose to buy at most one new item.

2.2 Solution Concepts

2.2.1 Stability

The key question in matching theory is whether or not an allocation A is stable, that is, whether or not there exists a blocking set of contracts Z such that all firms in Z_F will choose their contracts in Z from $Z \cup A$ (and possibly drop contracts in A).

Definition 4. An allocation A is **stable** if it is

1. **Individually rational:** for all $f \in F$, $C^f(A) = A|_f$.
2. **Unblocked:** There does not exist a nonempty **blocking set** $Z \subseteq X$ such that $Z \cap A = \emptyset$ and for all $f \in Z_F$, $Z|_f \subseteq C^f(A \cup Z)$.

Stability is inherently a price-theoretic notion. For an allocation A to be stable, we must be able to find *offer sets* (that is, sets of contracts offered to each firm) such that

- each contract in X is offered to some firm and
- each contract not part of the allocation A is rejected by every firm to which it is offered.¹⁷

¹⁷Here, we use the terminology “offer set” instead of “choice set” since firms are typically allowed to choose only one option (or point, or bundle) from a choice set; here they may choose any subset of contracts offered. (See the definition given on the first page of Chapter 1 of Mas-Colell et al. (1995) for instance.) Formally the choice set is the power set of the offer set.

These offer sets are similar to the price-theoretic notion of budget sets, in which prices define for each agent the set of bundles that are available to that agent. In the context of matching theory, market clearing means that a contract is chosen by its buyer if and only if it is also chosen by its seller. As in price theory, then, a matching-theoretic equilibrium (i.e. a stable allocation) is a set of bundles chosen by each agent (from his offer set) such that the market clears. This is similar to the idea of competitive equilibrium in price theory, where the demand for each item, given the prices, exactly equals the supply.¹⁸

Our stability notion is the natural generalization of the prior notions of stability in the one-to-one and many-to-one literature.¹⁹ As we discuss in the next section, this definition of stability is more stringent than the “chain stability” and “tree stability” concepts introduced by Ostrovsky (2008).

Stronger notions of stability (which we do not explore) allow for firms to play “strategically,” that is, to take on a set contracts Z from which they obtain a better overall allocation, even though a particular contract $z \in Z$ may not be part of $C^f(Z \cup A)$ for some f . However, allocations satisfying these stronger notions of stability, such as *setwise stability* (Sotomayor (1999), Westkamp (2010)) or *strong pairwise stability* (Echenique and Oviedo (2006)) often do not exist even for reasonable preferences.²⁰

2.3 Chain Stability

Ostrovsky (2008) introduced the following notion of *chains* and *chain stability*.

Definition 5. A set of contracts $\{x^1, \dots, x^N\}$ is a **chain** if

1. $x_B^n = x_S^{n+1}$ for all $n = 1, \dots, N - 1$.
2. $x_S^n = x_S^m$ implies that $n = m$.

¹⁸Hatfield et al. (2010) further develop this idea.

¹⁹See Roth and Sotomayor (1990) and Hatfield and Milgrom (2005) for definitions of stability in the one-to-one and many-to-one contexts. See Hatfield and Kominers (2010) for a discussion of the relationship between these concepts.

²⁰Additionally, we do not explore the solution concept of the *core* here, as it is known from an example of Blair (1984) that the core and the set of stable allocations are distinct even in the more restrictive setting of many-to-many matching (without contracts).

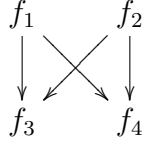


Figure 2: A contract structure for which chain stability need not imply stability.

3. $x_B^N \neq x_S^1$.

Definition 6. An allocation A is **chain stable** if it is individually rational and there is no chain that is a blocking set.

Intuitively, contracts form a chain if each firm holding a contract in the chain sells to the next firm in the chain (condition 1 of Definition 5), the chain never doubles back on itself (condition 2 of Definition 5), and the chain is not a loop, i.e. the buyer of the last contract is not the seller of the first (condition 3 of Definition 5). An allocation is chain stable if it is not blocked by chains of contracts.

Chain stability is not equivalent to the standard notions of stability used in the matching literature, which are special cases of our stability notion (see Roth and Sotomayor (1990)). Rather, chain stability is equivalent to *pairwise stability*, a much weaker concept. We later show (in Theorem 7) that chain stability is equivalent to stability with fully substitutable preferences and an acyclic contract set. However, the equivalence only holds when the contract set is acyclic and preferences are fully substitutable. If either condition fails, then the set of stable allocations may be a strict subset of the set of chain stable allocations and, furthermore, chain stability may be an intuitively unappealing solution concept.

For instance, consider the example where $F = \{f_1, f_2, f_3, f_4\}$ and

$$X = \{(i, j) \mid f_i, f_j \in F \text{ and } i \leq 2 < j\},$$

as depicted in Figure 2, and suppose firms have the following preferences:

$$P^{f_1} : \{(1, 3)\} \succ \{(1, 4)\} \succ \emptyset,$$

$$P^{f_2} : \{(2, 4)\} \succ \{(2, 3)\} \succ \emptyset,$$

$$P^{f_3} : \{(1, 3), (2, 3)\} \succ \emptyset,$$

$$P^{f_4} : \{(1, 4)\} \succ \{(2, 4)\} \succ \emptyset.$$

Note that the preferences of firm f_3 are not same-side substitutable. There does not exist a stable allocation; however, $\{(1, 4)\}$ is chain stable, as the only blocking set is $\{(1, 3), (2, 3)\}$, which—although seemingly a very natural deviation—is not a chain.

Similarly, when the contract set is not acyclic, it is also easy to construct examples where chain stability is strictly weaker than stability. For instance, consider the following example, where $x_S = y_B = f$, and $x_B = y_S = g$:

$$P^f : \{x, y\} \succ \emptyset,$$

$$P^g : \{x, y\} \succ \emptyset.$$

These preferences are fully substitutable; however, the contract set X is not acyclic, as it admits the cycle $\{x, y\}$. For this example, the set $\{x, y\}$ is the only stable allocation, while both $\{x, y\}$ and \emptyset are chain stable. The \emptyset allocation is an unappealing solution to this problem on both normative and positive grounds: $\{x, y\}$ is Pareto preferred to \emptyset , and it also seems unreasonable to consider a solution concept which does not allow f and g to take part in a joint deviation from \emptyset to $\{x, y\}$.

2.3.1 Strategy-proofness

We also consider strategic properties of matching mechanisms. A **matching mechanism** ψ is a mapping from the set of preference profiles to the set of allocations. In particular, we examine whether certain matching mechanisms are strategy-proof for some firms; that is, whether or not it is a weakly dominant strategy for firms to truthfully reveal their preferences.

Definition 7. A matching mechanism ψ is **strategy-proof** for $G \subseteq F$ if, for all $g \in G$ and any preference profile P , there is no preference profile P'_g such that $\psi(P'_g, P_{-g})$ is strictly preferred to $\psi(P)$ by g .²¹

Similarly, we can consider the incentives of groups of firms under a given matching mechanism.

Definition 8. A matching mechanism ψ is **group strategy-proof** for $G \subseteq F$ if, for all $H \subseteq G$ and any preference profile P , there is no preference profile P'_H such that $\psi(P'_H, P_{-H})$ is strictly preferred to $\psi(P)$ for all $h \in H$.

As is standard in the matching literature, for a matching mechanism to not be group strategy-proof, the deviation from truth-telling must make all firms in the coalition strictly better off.²²

3 Existence of Stable Allocations

3.1 Sufficiency of Fully Substitutable Preferences

In this section, we demonstrate that fully substitutable preferences are sufficient to guarantee the existence of a lattice of stable allocations when the contract set is acyclic, and for the standard opposition of interest results to hold. We give direct proofs of all the results in this section. In Section 2.3, we show that these results may be alternatively derived in an indirect fashion: we show (in Theorem 7) that when preferences are fully substitutable and the contract set is acyclic, chain stability and stability is equivalent, hence our existence and opposition results can be derived from corresponding results of Ostrovsky (2008) for chain stable allocations.

²¹Here, by P_{-g} we mean the preference profile of all agents excluding g . Similarly, P_{-G} is the preference profile of all agents except those in G .

²²See Hatfield and Kojima (2009) for a discussion of the motivation behind this definition.

To prove the existence of a stable allocation, we introduce the operator

$$\begin{aligned}\Phi_S(X^B, X^S) &\equiv X - R_B(X^B|X^S) \\ \Phi_B(X^B, X^S) &\equiv X - R_S(X^S|X^B) \\ \Phi(X^B, X^S) &= (\Phi_B(X^B, X^S), \Phi_S(X^B, X^S)),\end{aligned}$$

which generalizes the deferred acceptance algorithm of Gale and Shapley (1962). The inputs X^B and X^S are sets of contracts which, respectively, contain the options available to the firms as buyers, and the options available to the firms as sellers. At each iteration of Φ , we obtain a new set of seller options which includes all of X except $R_B(X^B|X^S)$, the set of contracts currently available to firms as buyers that they are rejecting. Similarly, the new set of buyer options is all of X except $R_S(X^S|X^B)$.

We first consider fixed points of the operator, and show that these fixed points correspond to stable allocations. Intuitively, at a fixed point we have that every contract is either being accepted (and hence is in both X^B and X^S), being rejected by a buyer (and hence is in X^B but not X^S) or being rejected by a seller (and hence is in X^S but not X^B). Since every contract not in $X^B \cap X^S$ is being rejected by some firm, there does not exist a blocking set of contracts Z such that each firm will desire all of those contracts, so long as preferences are fully substitutable.

Theorem 2. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then if $\Phi(X^B, X^S) = (X^B, X^S)$, the allocation $X^B \cap X^S$ is stable. Conversely, if A is a stable allocation, there exist $X^B, X^S \subseteq X$ such that $\Phi(X^B, X^S) = (X^B, X^S)$ and $X^B \cap X^S = A$.*

To see that full substitutability is necessary for stable allocations to generate fixed points, consider an example where $F = \{f_1, f_2\}$ and $X = \{x, y\}$, where $x_S = y_S = f_1$ and $x_B = y_B = f_2$, as shown in Figure 3. Now, consider the following preferences:

$$\begin{aligned}P^{f_1} &: \{x, y\} \succ \emptyset, \\ P^{f_2} &: \{x\} \succ \{y\} \succ \emptyset.\end{aligned}$$

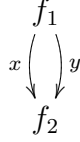


Figure 3: A simple contract structure where the fixed-point characterization fails if preferences are not fully substitutable.

Note that the preferences of f_1 are not substitutable. In this example, \emptyset is the unique stable allocation, and yet there do not exist (X^B, X^S) such that $X^B \cap X^S = \emptyset$ and $\Phi(X^B, X^S) = (X^B, X^S)$. If either x or y is in X^B , then one of these contracts is not rejected by the buyer f_2 , and hence this contract is in $X^S = \Phi_S(X^B, X^S) = X - R_B(X^B|X^S)$, contradicting the fact that $X^B \cap X^S = \emptyset$. If both x and y are in X^S , then neither is rejected by the seller f_1 , and so both are also in $X^B = \Phi_B(X^B, X^S) = X - R_S(X^S|X^B)$, again contradicting the fact that $X^B \cap X^S = \emptyset$.

To see that full substitutability is necessary for fixed points to imply stability, consider altering the example above so that preferences are given by

$$P^{f_1} : \{x, y\} \succ \emptyset,$$

$$P^{f_2} : \{x, y\} \succ \emptyset.$$

In this case, $\{x, y\}$ is the unique stable allocation, but $(\{x\}, \{y\})$ is a fixed point and corresponds to the allocation \emptyset .²³

Furthermore, acyclicity is necessary for fixed points to induce stable allocations. Consider the case where there are three firms, f_1, f_2, g , and the contract structure is as shown in Figure 4. When firms' preferences are given by

$$P^{f_1} : \{y, x^2\} \succ \{x^1, x^2\} \succ \emptyset, \quad P^{f_2} : \{x^2, x^1\} \succ \emptyset,$$

$$P^g : \{y\} \succ \emptyset,$$

²³Observe that \emptyset is chain stable in this example. Thus, note that focusing on stable allocations rather than chain stable allocations allows us to obtain a sharp fixed-point characterization: full substitutability is necessary for both directions of Theorem 2.

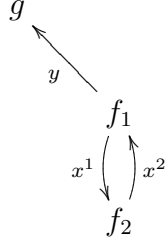


Figure 4: A contract cycle with an outside option.

no stable allocation exists. However, $(X^B, X^S) = (\emptyset, \{y, x^1, x^2\})$ is a fixed point of the operator.²⁴

Theorem 2 shows that, when preferences are fully substitutable, and the contract set is acyclic, there is a bijective correspondence between the set of fixed points of Φ and the set of stable allocations. We now define the order \sqsubseteq on $X \times X$ as

$$(X^B, X^S) \sqsubseteq (\hat{X}^B, \hat{X}^S) \text{ if } X^B \subseteq \hat{X}^B \text{ and } X^S \supseteq \hat{X}^S.$$

It is clear that Φ is isotone with respect to this order if preferences are fully substitutable.²⁵ Hence, by Tarski's theorem, there exists a lattice of fixed points of this operator. Furthermore, if the contract set is acyclic, these fixed points correspond to stable allocations.

Theorem 3. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then there exists a nonempty finite lattice of fixed points (X^B, X^S) of Φ which correspond to stable allocations $A = X^B \cap X^S$.*

²⁴It is not necessary that no stable allocation exist for fixed points to not correspond to stable allocations. Consider the case where $F = \{f_1, f_2\}$ and $X = \{x, y\}$, where $x_S = y_B = f_1$ and $x_S = y_B = f_2$, and preferences are given by:

$$\begin{aligned} P^{f_1} : \{x, y\} \succ \emptyset, \\ P^{f_2} : \{x, y\} \succ \emptyset, \end{aligned}$$

which are fully substitutable. $\{x, y\}$ is the unique stable allocation, but $(\{x, y\}, \emptyset)$ is a fixed point of $\Phi(\cdot)$ even though \emptyset is not a stable allocation.

²⁵This means that, when preferences are fully substitutable,

$$(X^B, X^S) \sqsubseteq (\hat{X}^B, \hat{X}^S) \Rightarrow \Phi(X^B, X^S) \sqsubseteq \Phi(\hat{X}^B, \hat{X}^S).$$

Furthermore, this lattice of fixed points has the same structure as in standard bilateral matching contexts. An allocation \hat{A} is the **buyer-optimal stable allocation** if it is preferred to all other stable allocations A by all firms who are exclusively buyers, that is $\hat{A} \succ_g A$ for all stable allocations A and for all $g \in \{f \in F \mid \nexists x \in X \text{ such that } x_S = g\}$. Since the set of the fixed points is a lattice, there exists a highest fixed point with respect to the order \sqsubseteq , which we denote by (\hat{X}^B, \hat{X}^S) . Since this fixed point has the largest set of contracts for the buyers to choose from, it is the unanimously most preferred stable allocation for all firms who are buyers but not sellers. Similarly, the lowest fixed point with respect to \sqsubseteq , denoted $(\check{X}^B, \check{X}^S)$, is the unanimously most preferred stable allocation for all firms who are sellers but not buyers.

Theorem 4. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then the highest fixed point (\hat{X}^B, \hat{X}^S) of Φ corresponds to the buyer-optimal stable allocation $\hat{X}^B \cap \hat{X}^S$, and the lowest fixed-point $(\check{X}^B, \check{X}^S)$ of Φ corresponds to the seller-optimal stable allocation $\check{X}^B \cap \check{X}^S$.*

Building upon machinery developed by Hatfield and Milgrom (2005), our proof of the existence of stable allocations naturally generalizes the deferred acceptance approach of Gale and Shapley (1962) to the context of matching with networks. Consequently, our existence (Theorem 3) and opposition of interest (Theorem 4) results naturally and directly generalize those of Gale and Shapley (1962) (for one-to-one and many-to-one matching), Roth (1984) (for many-to-many matching), Hatfield and Milgrom (2005) (for many-to-one matching with contracts), and Hatfield and Kominers (2010) (for many-to-many matching with contracts). Additionally, these results generalize the analogous results due to Echenique and Oviedo (2006) (for many-to-many matching) and Ostrovsky (2008) (for supply-chain matching).

3.2 Necessary Conditions for Stability

From the preceding analysis, it is clear that, in order to ensure the existence of a stable allocation, it is sufficient that preferences are fully substitutable and that the set of contracts is acyclic. In many-to-one matching with contracts, however, there exist weaker conditions on preferences that guarantee the existence of a stable allocation.²⁶ However, for the more general model of matching in networks, both conditions are necessary: If acyclicity fails, then there are fully substitutable preferences for each firm such that stable allocation exists. For acyclic contract sets, if one firm has preferences that are not fully substitutable, then there exist fully substitutable preferences for the other firms such that no stable allocation exists.

Theorem 5. *If the set of contracts X admits a cycle $L = \{x^1, \dots, x^N\}$ and there exists a firm $f \notin L_F$ and a contract between f and some firm in the cycle, then there exist fully substitutable preferences such that no stable allocation exists.*

An example of fully substitutable preferences where no stable match exists is given in the discussion of acyclicity in Section 3.1. Consider again the contract structure given in Figure 4. Intuitively, if contracts form a cycle (such as that between f_1 and f_2), then it is easy to construct fully substitutable preferences such that each firm wants either both contracts it is associated with in the cycle or no contracts at all. With these preferences, the empty allocation is not stable, as all members of the cycle agree that the contract cycle is better than nothing. However, if one member of the cycle (f_1) has an outside option as a buyer (y) then the cycle itself may not be stable as that one member may most prefer to choose this outside option as a buyer while keeping the contract where he is a seller (x^2) from the cycle. This implies that the cycle itself is unstable. However, any other allocation (such as $\{y, x^2\}$) is also unstable, as it is not individually rational for some agent. Hence,

²⁶Hatfield and Kojima (2008) show that substitutability is not necessary to guarantee existence of a stable allocation for many-to-one matching with contracts. Hatfield and Kojima (2010) and Hatfield and Kominers (2010) have found in that setting weaker conditions on preferences than substitutability that guarantee the existence of a stable allocation.

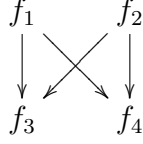


Figure 5: A contract structure in which stable allocations may not exist when preferences are not fully substitutable.

we see that even when preferences are reasonable, contract sets containing cycles may not admit stable allocations.

Furthermore, it is also necessary that the preferences of each firm be fully substitutable. Consider again the example from Section 2.3 (pictured in Figure 2), where $F = \{f_1, f_2, f_3, f_4\}$ and $X = \{(i, j) : f_i, f_j \in F \text{ and } i \leq 2 < j\}$, as pictured in Figure 5. Here, firms f_1 and f_2 are sellers, and firms f_3 and f_4 are buyers. Suppose that preferences are given by:

$$P^{f_1} : \{(1, 3)\} \succ \{(1, 4)\} \succ \emptyset,$$

$$P^{f_2} : \{(2, 4)\} \succ \{(2, 3)\} \succ \emptyset,$$

$$P^{f_3} : \{(1, 3), (2, 3)\} \succ \emptyset,$$

$$P^{f_4} : \{(1, 4)\} \succ \{(2, 4)\} \succ \emptyset.$$

It is not stable for both sellers f_1 and f_2 to sell to f_3 , as then f_2 would like to deviate and sell to f_4 , and f_4 would like to buy from f_2 . That, however, is also not stable, as then f_3 would buy nothing, and so f_1 would then sell to f_4 . This is also not stable, as then both sellers would prefer to sell to f_3 . Hence, no allocation is individually rational and unblocked. Generalizing this example, we obtain the following theorem.

Theorem 6. *Suppose X is exhaustive and acyclic, and there exists a firm f whose preferences are not fully substitutable, and there exist at least two firms upstream and two firms downstream of f . Then, there exist fully substitutable preferences for the firms other than f such that no stable allocation exists.²⁷*

²⁷Slightly weaker conditions on the number of firms other than f can be stated. However, these conditions are exceedingly technical; see the proof of Theorem 6 for details.

Theorem 6 is a generalization of the previous necessity results in the matching literature (see Hatfield and Kominers (2010)).

3.3 The Relationship between Stability and Chain Stability

It is immediate that, regardless of restrictions on preferences or the contract set, any stable allocation is chain stable.²⁸ However, as we now show, stability and chain stability coincide on domains where X is acyclic and preferences are fully substitutable.

Theorem 7. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then an allocation A is stable if and only if it is chain stable.*

Ostrovsky (2008) proved, under the assumption of fully substitutable preferences, that chain stability are equivalent to an alternative solution concept called *tree stability*.²⁹ Like chain stability, tree stability is weaker than stability, in general. However, stability and tree stability are equivalent under fully substitutable preferences; this fact is a simple corollary of Theorem 4 of Ostrovsky (2008) and our Theorem 7.

It follows from the preceding discussion that our Theorems 2–4 can be obtained indirectly using Theorem 7 and the Ostrovsky fixed-point characterization, existence, and opposition of interest results. However, there is no necessity result corresponding to our Theorem 6 in the Ostrovsky framework. To see this, consider a firm f with preferences of the form

$$P^f : \{x, y\} \succ \emptyset$$

where $x_B = y_B = f$, which are clearly not fully substitutable. A chain stable allocation exists for any fully substitutable preferences for other firms so long as the contract set X is acyclic. These chain stable allocations can be found by using the isotone operator Φ introduced above on the set $X - \{x, y\}$; these allocations will be chain stable as there are clearly (by Theorem 2) no blocking sets not involving firm f , and the only blocking

²⁸This follows as chain stability requires that blocking sets be chains, whereas our notion of stability puts no restrictions on the structure of the blocking set.

²⁹See Theorem 4 of Ostrovsky (2008).

set involving firm f could be $\{x, y\}$, which is not a chain. However, $\{x, y\}$ seems a very natural deviation.³⁰

4 The Structure of the Set of Stable Allocations

4.1 The Rural Hospitals Theorem

In the celebrated *rural hospitals theorem*, Roth (1986) proved that in a many-to-one (doctor-hospital) matching market with responsive preferences, any hospital that has unfilled positions at some stable matching is assigned exactly the same set of doctors at every stable matching.³¹ In the context of many-to-one (Hatfield and Milgrom (2005); Hatfield and Kojima (2010)) and many-to-many matching with contracts (Hatfield and Kominers (2010)), the rural hospitals theorem has been (partially) generalized to a statement regarding the number of contracts: in a many-to-many matching market with contracts, if preferences are substitutable and satisfy the law of aggregate demand, then every firm holds the same number of contracts in each stable allocation.³²

However, this statement is false in the context of matching in networks. For instance, consider the following simple set of preferences, where $F = \{f_1, f_2, f_3\}$ and

³⁰An explicit example of this is given in Section 2.3,

³¹Recall that hospital preferences are called **responsive** if they are consistent with a complete strict order over individual doctors.

³²Generalizations of the Roth (1986) rural hospitals theorem to the theory of matching with contracts have focused only upon the total number of contracts signed by each hospital h , since in general both the set of doctors with whom h contracts and the contract terms to which h agrees may vary across stable allocations. Indeed, Hatfield and Kojima (2010) give (in their Footnote 21) a many-to-one matching with contracts example in which there is a hospital h with unfilled positions such that both

- the set of doctors h contracts with, and
- the contract terms h receives from doctors

vary across stable allocations. Hence, the invariance of the number of contracts per firm across stable allocations seems to be the sharpest possible matching with contracts generalization of the Roth (1986) theorem.

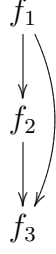


Figure 6: A contract structure in which the most immediate (putative) generalization of the rural hospitals theorem need not hold.

$X = \{(i, j) \mid f_i, f_j \in F \text{ and } i < j\}$, as pictured in Figure 6:

$$P^{f_1} : \{(1, 2)\} \succ \{(1, 3)\} \succ \emptyset,$$

$$P^{f_2} : \{(1, 2), (2, 3)\} \succ \emptyset,$$

$$P^{f_3} : \{(1, 3)\} \succ \{(2, 3)\} \succ \emptyset.$$

Note that firm f_2 has the simplest possible fully substitutable preferences where the set of contracts that firm f_2 signs as a buyer truly depends on the set of contracts she has access to as a seller. Furthermore, these preferences satisfy the laws of aggregate demand and supply. However, both $\{(1, 3)\}$ and $\{(1, 2), (2, 3)\}$ are stable allocations, and so we see that the number of contracts in different stable allocations may vary across stable allocations even when preferences satisfy the laws of aggregate supply and demand. Rather, the difference for each firm between the number of “buy” and “sell” contracts held is constant across the two stable allocations; it turns out this “balancing” property holds in general.

Theorem 8. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable and satisfy the laws of aggregate supply and demand. Then, for each firm, the difference between the number of contracts the firm buys and the number of contracts the firm sells is invariant across stable allocations.*

When all contracts are denoted in a fixed unit (as in exchange economies), Theorem 8 implies that each agent holds the same excess stock at every stable allocation. When contracts do not use a constant unit (as, for example, in labor markets in which both full- and

part-time job contracts are available), the exact numerical implications of Theorem 8 are less clear, as Hatfield and Kominers (2010) discuss. However, even in this case, the rural hospitals result is crucial for the strategy-proofness results of Section 4.2.

Note that Theorem 8 does generalize the prior rural hospitals theorems: in previous two-sided matching models, a firm f either only buys or only sells, hence Theorem 8 implies that f signs exactly the same number of contracts in every stable allocation. We have that the net position in contracts of a firm which both buys and sells is invariant across stable allocations, but the total number of contracts such a firm signs can vary across stable allocations. Furthermore, the laws of aggregate supply and demand are the weakest possible conditions that ensure this additional structure on the set of stable allocations.

Theorem 9. *Suppose that the set of contracts X is acyclic and exhaustive. If the preferences of some firm f do not satisfy the law of aggregate supply or the law of aggregate demand but are fully substitutable, then there exist preferences for the other firms satisfying full substitutability and the laws of aggregate supply and demand such that there exist two stable allocations such that the difference between the number of contracts f buys and the number of contracts f sells is different.*

4.2 Strategy-Proofness

In many-to-one matching with contracts, substitutes and the law of aggregate demand are enough to ensure that the buyer-optimal stable mechanism is strategy-proof for **unit demand** buyers, i.e. buyers who demand at most one contract. Unfortunately, in many-to-many contexts, it is not strategy-proof for either side of the market to reveal its preferences truthfully to any mechanism which chooses a stable allocation. However, even in the matching in networks framework, if some subset of firms acts only as buyers, and each of these buyers has unit demand, then the mechanism which chooses the buyer-optimal stable allocation will be strategy-proof for these buyers.³³

³³By symmetry, an analogous result applies to sellers.

Theorem 10. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable and satisfy the laws of aggregate supply and demand. If additionally, for all $g \in G \subseteq F$, the preferences of g exhibit unit demand, then any mechanism that selects the buyer-optimal stable allocation is (group) strategy-proof for G .*

Theorem 10 generalizes results of Dubins and Freedman (1981), Abdulkadiroğlu (2005), Hatfield and Milgrom (2005), and Hatfield and Kojima (2009); its proof follows exactly as in Hatfield and Kojima (2009). This result can be used to show a common corollary that the set of unit-demand buyers weakly prefers the buyer-optimal stable allocation to all other individually rational allocations.

Corollary 1. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable and satisfy the laws of aggregate supply and demand. If additionally, for all $g \in G \subseteq F$, the preferences of g exhibit unit demand, then there does not exist an individually rational allocation that every member of G strictly prefers to the buyer-optimal stable allocation.³⁴*

This result is commonly called “weak Pareto optimality” and was first shown by Roth (1982) for one-to-one matching; the most result general result to date is for many-to-one matching with contracts and is shown by Hatfield and Kojima (2009) and Kojima (2007).

5 Conclusion

In this paper we have extended the model of classical matching theory to consider networks of contracts. We have shown that, on the one hand, if the set of contracts is acyclic, and preferences are fully substitutable, not only do stable allocations exist, but they form a lattice. Moreover, classical results of matching theory, such as the rural hospitals and strategy-proofness theorems, generalize to this setting. On the other hand, Theorem 5 shows that, in the presence of fully substitutable preferences over contracts, stability cannot be guaranteed if there is a single cycle in the set of contracts; furthermore, Theorem 6

³⁴As with Theorem 10, the proof of Corollary 1 follows as in Hatfield and Kojima (2009).

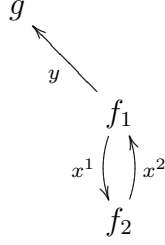


Figure 4: A contract cycle with an outside option.

shows that if even one firm does not have fully substitutable preferences, stability again cannot be guaranteed. Hence our work delineates a strict frontier for matching, in the sense that both acyclicity and full substitutability are both necessary and sufficient for classical matching theory approaches.

However, supplementing the set of contracts with a numeraire (over which utility is quasilinear) may allow us to go further. Consider the simplest example where a cyclic contract set confounds stability, drawn from the proof of Theorem 5. There are three firms, f_1, f_2, g , and the contract structure is as shown in Figure 4, which we reproduce here. When firms' preferences are given by

$$P^{f_1} : \{y, x^2\} \succ \{x^1, x^2\} \succ \emptyset, \quad P^{f_2} : \{x^2, x^1\} \succ \emptyset, \\ P^g : \{y\} \succ \emptyset,$$

there exists no stable allocation. Indeed, firm f_1 prefers its outside option y over x^1 , and hence the trade cycle $\{x^1, x^2\}$ always breaks down.

Problems of this form often arise in contracting relationships, and they have a well-known solution, albeit one outside the scope of classical matching theory. One resolution to this dilemma dates back to ideas of Vickrey (1961) and Pigou: firm f_1 should pay a transfer to f_2 equal to the value of the externality f_1 causes by dropping contract x^1 in favor of y . Generalizing this intuition, Hatfield et al. (2010) find that transferable utility promotes stability in some new settings.

Even with transferable utility, full substitutability is necessary in order to guarantee the

existence of stable allocations. However, many problems naturally generate complementarities; a hospital may open a new wing only if it acquires doctors of multiple specialities, or a firm may be able to operate more efficiently with more units.³⁵ Much work remains to be done to understand the dynamics and equilibria of matching markets with complementarities; we leave this topic for future research.

³⁵Milgrom (2007) and Day and Milgrom (2008) discuss some of the issues that arise with complementary preferences in package auctions. Klaus and Klijn (2005) discuss the “couples problem,” a type of complementary preferences that commonly arises in bilateral matching contexts.

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Appendix

Proof of Theorem 1

Proof of the “only if” Direction

Let the preferences of f be given by

$$X^N \succ X^{N-1} \succ \dots \succ X^2 \succ X^1 \succ \emptyset.$$

Consider any two offer vectors $\vec{q}^f(Y^q)$ and $\vec{r}^f(Y^r)$ such that $\vec{q} \leq \vec{r}$ and suppose that $\vec{q}_w^f = \vec{r}_w^f = -1$ for some w such that $w_B = f$. Suppose that

$$u(\vec{q}^f(\{w\} \cup Y^r)) - u(\vec{q}^f(Y^r)) > 0$$

then $w \in C^f(\{w\} \cup Y^r)$, so $w \in C^f(\{w\} \cup Y^q)$ due to full substitutability. Hence,

$$u(\vec{q}^f(\{w\} \cup Y^q)) - u(\vec{q}^f(Y^q)) > 0$$

and quasisubmodularity is satisfied. Hence, when we add any contract to the offer set as a buyer, quasisubmodularity is satisfied. (Clearly, when we add a contract to the offer set, the utility of the firm can not fall.)

Now suppose that $\vec{q}_w^f = \vec{r}_w^f = 0$ for some w such that $w_S = f$. Suppose that

$$u(\vec{q}^f(Y^q - \{w\})) - u(\vec{q}^f(Y^q)) < 0$$

then $w \in C^f(Y^q)$, so $w \in C^f(Y^r)$ by full substitutability. Hence,

$$u(\vec{q}^f(Y^r - \{w\})) - u(\vec{q}^f(Y^r)) < 0$$

and quasisubmodularity is satisfied when we remove any contract to the offer set as a seller. (Clearly, when we remove a contract from the offer set, the utility of the firm can not rise.)

Finally, note that it is enough to show it for adding (or removing) a single contract to Y^q and Y^r , since if quasisubmodularity fails for some $\vec{s} \geq 0$, it must fail for \vec{s}^f such that $\vec{s}_w^f \neq 0$ for only one contract.

Proof of the “if” Direction

Suppose that preferences violate condition #1 of same-side substitutability. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \notin C_B^f(Y \cup \{w\}) \text{ and } w \in C_B^f(\{x\} \cup Y \cup \{w\})$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u(\bar{q}^f(Y \cup \{w\})) - u(\bar{q}^f(Y)) = 0 < u(\bar{q}^f(\{x\} \cup Y \cup \{w\})) - u(\bar{q}^f(\{x\} \cup Y))$$

so u is not quasisubmodular.

Similarly, suppose that preferences violate condition #2 of same-side substitutability. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \notin C_S^f(Z \cup \{w\}) \text{ and } w \in C_S^f(\{x\} \cup Z \cup \{w\})$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u(\bar{q}^f(Y)) - u(\bar{q}^f(Y \cup \{w\})) = 0 > u(\bar{q}^f(Y)) - u(\bar{q}^f(Y \cup \{z\}))$$

so u is not quasisubmodular.

Now suppose that preferences violate condition #1 of cross-side complementarity. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \in C_B^f(Y \cup \{w\}) \text{ and } w \notin C_B^f(Y \cup \{z\})$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u(\bar{q}^f(Y \cup \{w\})) - u(\bar{q}^f(Y)) > 0 = u(\bar{q}^f(Y \cup \{w\})) - u(\bar{q}^f(Y \cup \{z\}))$$

so u is not quasisubmodular.

Similarly, suppose that preferences violate condition #2 of cross-side complementarity.

Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \in C_S^f(Z \cup \{w\}) \text{ and } w \notin C_S^f(Z \cup \{w\} \cup Y)$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u(\bar{q}^f(Y)) - u(\bar{q}^f(Y)) > 0 = u(\bar{q}^f(\{x\} \cup Y)) - u(\bar{q}^f(\{x\} \cup Y))$$

so u is not quasimodular.

Proof of Theorem 2

First Part

Suppose that $X^B \cap X^S \equiv A$ is a fixed point, but that A is either not individually rational or admits a blocking set Z . If A is not individually rational, there must exist $x \in A$ such that $x \in R^f(A)$ for some $f \in F$. Then either $x \in R_B^f(A|A)$ and $x_B = f$ or $x \in R_S^f(A|A)$ and $x_S = f$. Assume the former. (The latter case is symmetric.) Then $x \in R_B^f(X^B|A)$ by same-side substitutes. However, every contract in the set $X^S - A$ is rejected by some firm as a seller, and so $R_B^f(X^B|A) = R_B^f(X^B|X^S)$ by individual rationality. Hence $x \in R_B^f(X^B|X^S)$, and hence $x \notin X^S$, and hence $x \notin X^B \cap X^S = A$, a contradiction.

If there exists a blocking set Z for A , consider a contract z such that $z_S \trianglelefteq y_S$ for all other $y \in Z$.³⁶ By same-side substitutes, since $z \in C_S^{zS}(Z \cup A|Z \cup A)$, we have that $z \in C_S^{zS}(\{z\} \cup A|Z \cup A) = C_S^{zS}(\{z\} \cup A|A)$ as there are no contracts in Z such that z_S is a buyer by assumption. Hence, by individual rationality, $z \in C_S^{zS}(\{z\} \cup X^S|A)$ and then by cross-side complements, $z \in C_S^{zS}(\{z\} \cup X^S|X^B)$. Hence, if $z \in X^S$, then it would also be in $X^B = X - R_S(X^S|X^B)$. But $z \notin A = X^B \cap X^S$ by assumption, and $X^B \cup X^S = X$, and so $z \in X^B$. Now consider an arbitrary contract $w \in Z$, and suppose that for all contracts $y \in Z$ such that $y_S \triangleleft w_S$, $y \in X^B$. By same-side substitutes, since $w \in$

³⁶Recall that acyclicity guarantees there exists an order \triangleleft on F such that $x_S \triangleleft x_B$ for all $x \in X$.

$C_S^{ws} (Z \cup A | Z \cup A)$, we have that $w \in C_S^{ws} (\{w\} \cup A | Z \cup A)$. Now, by induction, for any contract $y \in Z$ such that $y_B = w_S$, $y \in X^B$. Hence, $\{y \in Z | y_B = w_S\} \subseteq X^B$, and $A \subseteq X^B$, implying that $\{y \in Z | y_B = w_S\} \cup A \subseteq X^B$, and so by cross-side complementarity, $w \in C_S^{ws} (\{w\} \cup A | X^B)$. Therefore, by individual rationality, $w \in C_S^{ws} (\{w\} \cup X^S | X^B)$. Thus, if $w \in X^S$, then it would also be in $X^B = X - R_S (X^S | X^B)$. But $w \notin A = X^B \cap X^S$ by assumption, and $X^B \cup X^S = X$, and so $w \in X^B$. Using induction then, we have that $Z \subseteq X^B$. Working symmetrically for buyers, we have that $Z \subseteq X^S$. Hence, $Z \subseteq X^S \cap X^B = A$ and hence Z is not a blocking set, a contradiction.

Second Part

Suppose A is a stable allocation. We construct X^B and X^S iteratively over firms; since contracts are acyclic, we order the firms f_1, \dots, f_N so that $f_n \triangleleft f_{n+1}$ for all $n = 1, \dots, N-1$.

1. Let $X^B(0) = X^S(0) = A$. Let

$$\begin{aligned} X^B(n) &= \{x \in (X - X^S(n-1)) | x_B = f_n\} \cup X^B(n-1) \\ X^S(n) &= \left\{x \in X | x \in R_S^{f_n}(\{x\} \cup A | X^B(n))\right\} \cup X^S(n-1) \end{aligned}$$

Finally, let $X^B = X^B(N)$ and $X^S = X^S(N)$.

We now show that (X^B, X^S) is a fixed point. We have

$$\begin{aligned} X - R_S(X^S | X^B) &= X - \bigcup_{n=1}^N R_S^{f_n}(X^S | X^B) \\ &= X - \bigcup_{n=1}^N R_S^{f_n}(X^S | X^B(n)) \text{ as } X^B(n) = \{x \in X^B | x_B = f_n\} \\ &= X - \bigcup_{n=1}^N R_S^{f_n}(X^S(n) | X^B(n)) \text{ as } X^S(n) = \{x \in X^S | x_S = f\} \\ &= X - \bigcup_{n=1}^N R_S^{f_n}(X_S(n) - A | f_n) \text{ as } f_n \text{ has fully substitutable preferences} \\ &= X - (X^S - A) \text{ by the definition of } X^S \\ &= X^B \text{ by the definition of } X^B \end{aligned}$$

Similarly,

$$\begin{aligned}
X - R_B (X^B | X^S) &= X - \bigcup_{n=1}^N R_B^f (X^B | X^S) \\
&= X - \bigcup_{n=1}^N R_B^f (X^B | X^S (n)) \text{ as } X^S (n) = \{x \in X^S | x_S = f_n\} \\
&= X - \bigcup_{n=1}^N R_B^f (X^B (n) | X^S (n)) \text{ as } X^B (f) = \{x \in X^B | x_B = n\} \\
&= X - \bigcup_{n=1}^N (X^B (n) - A|_{f_n}) \text{ as shown below} \\
&= X - (X^B - A) \text{ by the definition of } X^B \\
&= X^S \text{ by the definition of } X^S
\end{aligned}$$

To see the fourth equality, suppose that there exists a nonempty set of contracts

$$Y \equiv \{y^1, \dots, y^k\} \in \bigcup_{n=1}^N (X^B (n) - A|_{f_n}) - \bigcup_{n=1}^N R_B^f (X^B (n) | X^S (n))$$

We also know that no contract Y is rejected by a seller (assuming they have access to X^B as buyers) as these are contracts X^B . Hence, Y is a blocking set and hence A is not stable, a contradiction. Finally, $A|_{f_n} \cap R_B^f (X^B (n) | X^S (n)) = \emptyset$ as A is individually rational.

Finally, we need to show that $X^B \cap X^S = A$. First, since $X^B (0) \cap X^S (0) = A$ and $X^B (n-1) \subseteq X^B (n)$ and $X^S (n-1) \subseteq X^S (n)$, $A \subseteq X^B (0) \cap X^S (0)$. Suppose that $z \in X^S - A$. Then $z \notin X^B$, as it could only be added in the z_B -th step and since $z \in X^S$, $a \in X^S (f_{z_B} - 1)$.

Proof of Theorem 4

For any stable allocation (X^B, X^S) , we have that

$$\begin{aligned}
X^B \cap X^S &= X^B \cap (X - R_B (X^B | X^S)) \\
&= X^B \cap (X - (X^B - C_B (X^B | X^S))) \\
&= X^B - (X^B - C_B (X^B | X^S)) \\
&= C_B (X^B | X^S)
\end{aligned}$$

Since for each firm f who is only a buyer, $C_B^f (X^B | X^S) = C^f (X^B)$, the firm f has a strictly larger choice set under (\hat{X}^B, \hat{X}^S) than under any other stable allocation, and

hence (weakly) prefers (\hat{X}^B, \hat{X}^S) . The proof that $(\check{X}^B, \check{X}^S)$ is the seller-optimal stable allocation is symmetric.

Proof of Theorem 5

Let y be the contract with f and some member of the cycle. Define $f_n \equiv x_S^n$. Suppose that $y_S = f_1$ and $y_B = g$. (The other case is symmetric.) Then for $n = 2, \dots, N$, let the preferences of firm f_n be

$$P^{f_n} : \{x^n, x^{n+1}\} \succ \emptyset$$

which satisfies same-side substitutes and cross-side complements. Let

$$P^g : \{y\} \succ \emptyset$$

$$P^{f_1} : \{y, x^N\} \succ \{x^1, x^N\} \succ \emptyset$$

which satisfies full substitutability; let all other firms desire no contracts. Any set $Y \not\subseteq L \cup \{y\}$ is not stable, as it is not individually rational. Any $Z \subsetneq L \cup \{y\}$ is not stable, as it is not individually rational unless $Z = \emptyset$, in which case L is a blocking set, or $Z = L$, in which case $\{y\}$ is a blocking set.

Proof of Theorem 6

If the preferences of a firm f do not satisfy same-side substitutes, then there exist contracts x, y and sets of contracts Y, Z such that

$$y \notin C_B^f(Y|Z) \text{ but } y \in C_B^f(\{x\} \cup Y|Z) \text{ or}$$

$$y \notin C_S^f(Y|Z) \text{ but } y \in C_S^f(Y|\{x\} \cup Z)$$

Assume the former; the latter case is symmetric. There are two cases.

Case 1: $x_S \neq y_S$. Let the firms in Y_S and Z_B (except x_S and y_S) have preferences such that they are willing to accept any all contracts with f that they are associated with.

Let y_S have preferences such that he would be willing to accept any and all of the

contracts he is associated with in Y , except that he wants exactly one of w and y , and would prefer y , where $w_S = y_S$ and $w_B = x_B$. Let x_S have preferences such that he would be willing to accept any and all of the contracts he is associated with in Y , except that he also desires w and only desires x if he obtains w . Finally, let all other firms not want any contracts. Now suppose that there is a stable allocation A .

1. Suppose $A|_f \prec_f C^f(Y \cup Z)$. If A is individually rational, all other firms want their contracts in $C^f(Y \cup Z)$ irrespective of their other contracts, so $C^f(Y \cup Z)$ is a blocking set.
2. Suppose $A|_f = C^f(Y \cup Z)$. Then both x_S and y_S desire the contract w ; hence $w \in A$. But then $C^f(\{x\} \cup Y \cup Z)$ is a blocking set.
3. Suppose $C^f(\{x\} \cup Y \cup Z) \succ_f A|_f \succ_f C^f(Y \cup Z)$. In this case, $x \in A$ as otherwise $A|_f$ is available to f from $Y \cup Z$, so we could not have $A|_f \succ_f C^f(Y \cup Z)$. But then, $C^f(\{x\} \cup Y \cup Z)$ is a blocking set.
4. Suppose $C^f(\{x\} \cup Y \cup Z) = A|_f$. Then if $w \in A$, A is not individually rational for y_S ; if $w \notin A$, A is not individually rational for x_S .

Case 2: $x_S = y_S \equiv d$. Let the firms in Y_S and Z_B (except y_S) have preferences such that they are willing to accept any all contracts with f that they are associated with. By assumption, there are two firms, g and h , downstream of f , and one firm, e , upstream of f , and hence upstream of g and h . Now consider the contracts v, v', w and w' such that $v_S = w_S = d$, $v'_S = w'_S = e$, $v_B = v'_B = g$ and $w_B = w'_B = h$ (which exist as X is exhaustive). g and h are willing to accept any and all contracts in Y , and outside of that, have preferences given by

$$P^g : \{v\} \succ \{v'\} \succ \emptyset$$

$$P^h : \{w'\} \succ \{w\} \succ \emptyset$$

and e is always willing to accept any and all contracts in Y , and outside of that, has preferences

$$P^e : \{v'\} \succ \{w'\} \succ \emptyset$$

Finally, d is always willing to accept any and all contracts in Y except y ; his preferences over x, y, w, v are responsive, where he wishes to obtain two contracts in accord with the preferences over singleton contracts given by $w \succ z \succ x \succ v$. Now suppose there exists a stable allocation A .

1. Suppose $A|_f \prec_f C^f(Y \cup Z)$. Then $C^f(Y \cup Z)$ constitutes a blocking set, as all the firms want their contracts in $C^f(Y)$, irrespective of other contracts.
2. Suppose $A|_f = C^f(Y \cup Z)$. Since d does not obtain x or y , he desires both v and w . For A to be stable then, d obtains v . Furthermore, since e does not obtain v' , for A to be stable, e must obtain w' . Hence $\{w', v\} \subseteq A$ and $w \notin A$. Hence $C^f(\{x\} \cup Y \cup Z)$ is a blocking set.
3. Suppose $C^f(Y \cup Z) \prec_f A|_f \prec C^f(\{x\} \cup Y \cup Z)$. Then $x \in A$, so $C^f(\{x\} \cup Y \cup Z) - \{x\}$ is a blocking set, as d will always take y and the other firms in Y and Z will always accept offers of any and all contracts in Y .
4. Suppose $C^f(\{x\} \cup Y \cup Z) = A|_f$. Then $v' \in A$, as otherwise $\{v'\}$ is a blocking set. (Note that in this case $v \notin A$, as then $\{x, y, v\} \subseteq A$, and so A is not individually rational for d .) But $v' \in S$ implies that $w' \notin S$, by the individual rationality of e . Hence w is a blocking set. (Note that $w \notin A$, as then $\{w, y, w\} \subseteq A$, and so A is not individually rational for d .)

If the preferences of a firm f do not satisfy cross-side complements, then there exists contracts y, z and sets of contracts Y, Z such that

$$y \in C_B^f(\{y\} \cup Y|Z) \text{ but } y \notin C_B^f(\{y\} \cup Y|\{z\} \cup Z) \text{ or}$$

$$z \in C_S^f(\{z\} \cup Z|Y) \text{ but } z \notin C_S^f(\{z\} \cup Z|\{y\} \cup Y)$$

Assume the latter; the former case is symmetric.

Let the firms in Y_S and Z_B (except y_S and z_B) have preferences such that they are willing to accept any all contracts with f that they are associated with. Let y_S have preferences such that he would be willing to accept any and all of the contracts he is associated with in Y , except that he wants exactly one of w and y , and would prefer w , where $w_S = y_S$ and $w_B = z_B$. Let z_B have preferences such that he would be willing to accept any and all of the contracts he is associated with in Z , except that he wants exactly one of w and z , and would prefer z . Finally, let all other firms not want any contracts. Now suppose that there is a stable allocation A .

1. Suppose $A|_f \prec_f C^f(Y \cup \{z\} \cup Z)$. If A is individually rational, all other firms want their contracts in $C^f(Y \cup \{z\} \cup Z)$ irrespective of their other contracts, so $C^f(Y \cup \{z\} \cup Z)$ is a blocking set.
2. Suppose $A|_f = C^f(Y \cup Z)$. Then both y_S and z_B desire the contract w ; hence $w \in A$. But then $A \supseteq \{w\} \cup C^f(Y \cup \{z\} \cup Z)$ is not individually rational for y_S .
3. Suppose $C^f(\{y\} \cup Y \cup \{z\} \cup Z) \succ_f A|_f \succ_f C^f(Y \cup \{z\} \cup Z)$. In this case, $y \in A$ as otherwise $A|_f$ is available to f from $Y \cup \{z\} \cup Z$, so we could not have $A|_f \succ_f C^f(Y \cup \{z\} \cup Z)$. But then, $C^f(\{x\} \cup Y \cup \{z\} \cup Z)$ is a blocking set.
4. Suppose $C^f(\{y\} \cup Y \cup \{z\} \cup Z) = A|_f$. Then if $w \in A$, A is not individually rational for z_B ; if $w \notin A$, $\{w\}$ is a blocking set.

Proof of Theorem 7

Consider an allocation A that is not stable. If it is not individually rational, then it is not chain stable by definition. Hence, suppose there is a blocking set Z . Since X is acyclic, there is an ordering of firms in Z_F such that $x_B \triangleright x_S$. Find the firm f for which $f \triangleright g$ for all $g \in Z_F$, and consider one contract $y^1 \in Z$ such that $y_B^1 = f$. Now consider y_S^1 . By same-side substitutes, $y^1 \in C_B^{y_B^1}(\{y^1\} \cup A|A)$ and $y^1 \in C_S^{y_S^1}(\{y^1\} \cup A|A \cup Z)$.

If $y^1 \in C_S^{y^1}(\{y^1\} \cup A|A)$, then $\{y^1\}$ is a chain and a blocking set and we are done, if not there exists a contract $y^2 \in Z$ such that $y^1 \in C_S^{y^1}(\{y^1\} \cup A|A \cup \{y^2\})$ and $y^2 \in C_B^{y^1}(\{y^2\} \cup A|A \cup \{y^1\})$, by same-side substitutes. (If no such y^2 exists, then preferences do not satisfy same-side substitutes, as there exists $y^2 \notin C_B^{y^1}(\{y^2\} \cup A|A \cup \{y^2\})$, but $y^2 \in C_B^{y^1}(Z \cup A|A \cup \{y^2\})$.) Using the same arguments for the contract y^2 , either $\{y^1, y^2\}$ is a chain and a blocking set, or there exists $y^3 \in Z$ such that $y^2, y^3 \in C_S^{y^2}(\{y^1, y^2, y^3\} \cup A)$. More generally, $\{y^1, y^2, \dots, y^n\}$ is a chain and a blocking set, or there exists $y^{n+1} \in Z$ such that $y^n, y^{n+1} \in C_S^{y^n}(\{y^1, \dots, y^{n+1}\} \cup A)$. Since X is finite, Z is finite, and hence there exists a chain $\{y^1, \dots, y^N\}$ that is a blocking set.

5.1 Proof of Theorem 8

Consider any stable allocation A associated with the fixed point (X^S, X^B) and the seller-optimal stable allocation \hat{A} associated with the fixed point (\hat{X}^S, \hat{X}^B) . Consider an arbitrary firm f . We have that

$$\begin{aligned} \left| C_f^S(\hat{X}^S|\hat{X}^B) \right| - \left| C_f^B(\hat{X}^B|\hat{X}^S) \right| &\geq \left| C_f^S(X^S|\hat{X}^B) \right| - \left| C_f^B(\hat{X}^B|X^S) \right| \\ &\geq \left| C_f^S(X^S|X^B) \right| - \left| C_f^B(X^B|X^S) \right|, \end{aligned}$$

where the first inequality follows by the LoAS, as $\hat{X}^S \supseteq X^S$, and the second follows by the LoAD, as $\hat{X}^B \subseteq X^B$. Hence, the difference between the number of contracts f sells and the number f buys is weakly greater under (\hat{X}^S, \hat{X}^B) than (X^S, X^B) . However, summing over all firms, the difference between the number of contracts bought and the number of contracts sold is zero. Hence, for each firm the change in the difference between the number of sold and the number bought is zero.

Proof of Theorem 9

Let the ordering of firms given X such that no firm sells to a lower-ordered firm by \triangleleft . There are three possibilities:

1. There exists a contract $z \in X$ such that

$$x, y \in C_S^f(Z|Y) \text{ and } x, y \notin C_S^f(\{z\} \cup Z|Y)$$

(The case where f is a buyer is symmetric.) For any firm $g \in (Y \cup Z) - \{x_B, y_B, z_B\}$ let them desire any and all contracts in $Y \cup Z$ they are associated with. Furthermore, let $\{x_B, y_B, z_B\}$ all desire any and all contracts they are associated with in Z regardless of their other contracts.

There are two subcases:

- (a) We have either $f \triangleleft x_B \triangleleft z_B$ or $f \triangleleft y_B \triangleleft z_B$; assume the former (the latter is symmetric). Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z - \{x, y, z\}$:

$$P^{x_B} : \{x, \hat{y}\} \succ \{x, \hat{z}\} \succ \{x\} \succ \{\hat{y}\} \succ \{\hat{z}\} \succ \emptyset$$

$$P^{y_B} : \{y\} \succ \{\hat{y}\} \succ \emptyset$$

$$P^{z_B} : \{\hat{z}\} \succ \{z\} \succ \emptyset$$

There are two stable allocations: $C^f(Z \cup Y) \cup \{\hat{z}\}$ and $C^f(\{z\} \cup Z \cup Y) \cup \{\hat{y}\}$, and hence the conclusion of the rural hospitals theorem fails for f .

- (b) We have $f \triangleleft z_B \triangleleft x_B, y_B$. Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z - \{x, y, z\}$:

$$P^{z_B} : \{z, \hat{x}\} \succ \emptyset$$

$$P^{y_B} : \{y\} \succ \emptyset$$

$$P^{x_B} : \{x\} \succ \{\hat{x}\} \succ \emptyset$$

There are two stable allocations: $C^f(Z \cup Y)$ and $C^f(\{z\} \cup Z \cup Y) \cup \{\hat{x}\}$, and hence the conclusion of the rural hospitals theorem fails for f .

2. There exists a contract $z \in X$ such that

$$x \in C_S^f(Z|Y), y \notin C_B^f(Y|Z) \text{ and } x \notin C_S^f(\{z\} \cup Z|Y), y \in C_B^f(Y|\{z\} \cup Z)$$

(The case where f is a buyer is symmetric.) For any firm $g \in (Y \cup Z) - \{x_B, y_B, z_B\}$ let them desire any and all contracts in $Y \cup Z$ they are associated with. Furthermore, let $\{x_B, y_B, z_B\}$ all desire any and all contracts they are associated with in Z regardless of their other contracts. Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z \cup Y - \{x, y, z\}$:

$$P^{x_B} : \{x\} \succ \emptyset$$

$$P^{y_B} : \{y\} \succ \{\hat{y}\} \succ \emptyset$$

$$P^{z_B} : \{\hat{y}\} \succ \{z\} \succ \emptyset$$

There are two stable allocations: $C^f(Z \cup Y) \cup \{\hat{y}\}$ and $C^f(\{z\} \cup Z \cup Y)$, and hence the conclusion of the rural hospitals theorem fails for f .

3. There exists a contract $z \in X$ such that

$$x, y \notin C_B^f(Y|Z) \text{ and } x, y \in C_B^f(Y|\{z\} \cup Z)$$

(The case where f is a buyer is symmetric.) For any firm $g \in (Y \cup Z) - \{x_B, y_B, z_B\}$ let them desire any and all contracts in $Y \cup Z$ they are associated with. Furthermore, let $\{x_B, y_B, z_B\}$ all desire any and all contracts they are associated with in Z regardless of their other contracts. Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z \cup Y - \{x, y, z\}$:

$$P^{x_B} : \{x\} \succ \emptyset$$

$$P^{y_B} : \{y\} \succ \{\hat{y}\} \succ \emptyset$$

$$P^{z_B} : \{\hat{y}\} \succ \{z\} \succ \emptyset$$

There are two stable allocations: $C^f(Z \cup Y) \cup \{\hat{y}\}$ and $C^f(\{z\} \cup Z \cup Y)$, and hence the conclusion of the rural hospitals theorem fails for f .