Multilateral Matching☆

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Abstract

We introduce a matching model in which agents engage in joint ventures via multilateral contracts. This approach allows us to consider production complementarities previously outside the scope of matching theory. We show analogues of the first and second welfare theorems, and, when agents' utilities are concave in venture participation, show that competitive equilibria exist, correspond to stable outcomes, and yield core outcomes. Competitive equilibria exist in our setting even when externalities are present.

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1. Introduction

Multiparty business enterprises take a variety of forms: manufacturing requires complementary inputs for production; consumer products firms coordinate advertising campaigns across multiple publishers in order to ensure that each consumer is exposed to multiple advertisements[1]; information technology firms collaborate on joint research ventures; cinema productions' actors are concerned with the identities of their costars and directors. In all of these settings, agents' preferences exhibit a form of complementarity: the willingness of two agents to contract with each other may be contingent on those agents' abilities to contract with third parties.

A natural equilibrium notion for multiparty contracting settings is matching-theoretic *stability*, the requirement that no set of agents can profitably recontract. Unfortunately, when agents contract over discrete goods or services, stable outcomes do not necessarily exist in the presence of complementarities across contracts. Consequently, standard matching theory typically rules out all forms of contractual complementarity, and can not be used to study multiparty enterprises.

This paper introduces a novel matching model with transferable utility in which sets of two or more agents may enter into multilateral contracts. Certain forms of complementarity can be expressed through such contracts; in particular, our model embeds a large class of economies with production complementarities. Our key insight is that stable multilateral contracting outcomes *do* exist when agents contract over continuously divisible quantities, so long as agents' valuations over production and consumption are concave.[2,3] Conversely, we also show a maximal domain result: If any one agent's valuation is not concave, then the existence of competitive equilibria can not be guaranteed. Furthermore, when agents' utilities are concave, stable outcomes directly correspond to competitive equilibria. Conversely, competitive equilibria induce outcomes that are strongly group stable and in the core.[4,5] Analogues of the first and second welfare theorems hold as well, showing in particular that stable outcomes (and competitive equilibria) are efficient. While our basic model disallows contractual externalities, competitive equilibria continue to exist even when such externalities are introduced (although they may not be efficient).

Previous work in matching theory has required (either explicitly or implicitly) that agents interact via bilateral contractual relationships[6]; in medical labor markets, medical students

[1] We thank Preston McAfee for suggesting this example, which is particularly relevant in the sale of Internet display advertisements.

[2] The assumption of concavity is natural in settings with decreasing returns to scale and scope. However, it is violated in settings with fixed costs or increasing returns to scale.

[3] Note that our model admits two different types of complementarities: those that can be expressed through multilateral ventures, as well as those allowed by the class of concave valuation functions.

[4] Note that the correspondence between stable outcomes, core outcomes, and competitive equilibria justifies our attention to the competitive equilibrium solution concept, despite the presence of personalized prices in our setting.

[5] Hatfield et al. [23] obtain analogous results in a setting distinct from ours, in which agents trade via discrete, bilateral contracts. It is known that analogous results do not hold in matching settings without transfers (Echenique and Oviedo [14], Klaus and Walzl [26]).

[6] For example, bilateral structure is imposed on relationships in the models of Gale and Shapley [19], Crawford and Knoer [13], Kelso and Crawford [25], Roth [32], Hatfield and Milgrom [24], Echenique and

"sell" their services to hospitals (Roth and Peranson [33]), and in school choice applications, schools "sell" their services to students (Balinski and Sönmez [10], Abdulkadiroğlu and Sönmez [4], Abdulkadiroğlu, Pathak, and Roth [1, 2], Abdulkadiroğlu, Pathak, Roth, and Sönmez [3]). The restriction to bilateral contracts was material in the previous work as, in order to guarantee the existence of equilibria, agents were required to view contracts as substitutes (see Hatfield et al. [23] and references contained therein). Meanwhile, it is well-known that equilibria may not exist in discrete matching models with multilateral contracting (see Alkan [5] and Chapter 2 of Roth and Sotomayor [34]). Previous work has shown the existence of equilibria in models with discrete contracts and complementarities by assuming a very large number of agents (Ellickson et al. [15, 16], Azevedo et al. [9], Azevedo and Hatfield [8]). By contrast, we find equilibria—irrespective of the number of agents—by allowing contract participation to vary continuously.[7,8]

The presence of continuously divisible contracts makes our underlying model similar to models of general equilibrium (Arrow and Debreu [6], Mas-Colell [29], Mas-Colell et al. [30]). However, unlike in standard general equilibrium theory, we consider production (and other) relationships that are agent-specific: in our framework, a set of agents may share a nonpublic production technology, and that technology may require inputs from specific agents, such as human capital.[9,10] Notwithstanding, we do not strictly extend general equilibrium theory, as we impose the requirement that agents' utilities be quasilinear in a numeraire.

The remainder of the paper is organized as follows. In the next section, we illustrate our model with a simple example (concrete production). In Section 3, we present our model in generality. We prove welfare theorems and existence results for competitive equilibria in Section 4; we then analyze the relationship between competitive equilibria, stable outcomes, and the core in Section 5. In Section 6, we present an application: economies with production complementarities embed naturally into the multilateral matching framework. We present a second application in Section 7, showing that the multilateral matching framework can be used to prove the existence of competitive equilibria in the continuum-of-agents setting of Azevedo et al. [9]. We then extend the multilateral matching framework to include contractual externalities in Section 8. We conclude in Section 9. All proofs are presented in

Oviedo [14], Ostrovsky [31], Hatfield and Kominers [22], and Hatfield et al. [23].

[7] In particular, our framework is conceptually distinct from the setting of the clubs literature (Ellickson et al. [15, 16]). In our work, we impose no structure on the set of agents, but require that joint venture participation levels are divisible, while in the clubs literature, participation in a club is a binary decision, but markets are required to be large (and agents are required to be of distinct types).

[8] However, as we show in Section 7, it is possible to derive the main result of Azevedo et al. [9] (in the finite type case) as a direct consequence of our competitive equilibrium existence theorem for multilateral matching economies.

[9] As we show in Appendix A, in the special case where agents' valuations satisfy a monotonicity condition, our existence result can be re-derived by constructing a corresponding general equilibrium model with agent-specific inputs and outputs of production.

[10] For production processes with complementary inputs, it is possible to model a multilateral contract as a collection of bilateral contracts, as we illustrate Section 6. However, for settings with externalities across contractual partners, such as joint research ventures and entertainment production, multilateral contracting can not be reduced to a model with only bilateral contracting. (To see why the entertainment industry requires multilateral contracting, note that actors contract with studios, but face externalities derived from the studio's choices of other actors for a given production.)

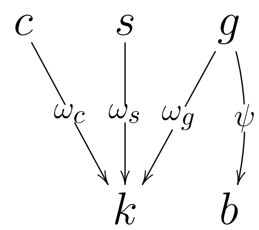

Figure 1: The example economy.

Appendix B.

2. An Illustrative Example

We illustrate our approach with a concrete example, using multilateral matching to model ready-mix concrete production.[11,12] Ready-mix concrete is produced by mixing three complementary inputs—cement, gravel, and sand—in proportions of approximately 1:2:2.[13] All three of these inputs are expensive to transport because of their weights; thus, each input good is only sold locally through relationship-specific contracts that incorporate transport costs (Syverson [36]).

The presence of input complementarities renders concrete production outside the scope of previous matching models. Indeed, previous work has required input substitutability in order to guarantee equilibrium existence (Gul and Stacchetti [20], Hatfield et al. [23]). As we illustrate, requiring continuous production adjustment (instead of allowing discrete adjustment as in the previous literature) enables us to relax the substitutability requirement and study industries, such as concrete production, with input complementarities.

It is natural to model the supply structure of a concrete producer k as requiring bilateral relationships ω_c, ω_g, and ω_s for the sale of cubic yards of cement, gravel, and sand, respectively, with suppliers c, g, and s. The gravel supplier g also has an outside option, ψ, to sell to another buyer, b. This economy structure is depicted in Figure 1.

Assuming constant marginal costs of cement and gravel production, and an increasing marginal cost of sand production, we assume the following supplier *valuation functions*:

$$\begin{aligned} v^c(r_{\omega_c}) &= -80 r_{\omega_c}, \\ v^g(r_{\omega_g}, r_\psi) &= -25(r_{\omega_g} + r_\psi), \\ v^s(r_{\omega_s}) &= -5 r_{\omega_s} - \frac{1}{16}(r_{\omega_s})^2, \end{aligned}$$

[11] In principle, this example can be studied using only bilateral contracts (as noted in Footnote 10); however, using the multilateral matching framework greatly simplifies the analysis. Note also that this example does not use the full generality of our framework—multilateral matching can be used to study economies with externalities across contractual parties, which can not be embedded into bilateral contracting models.

[12] In addition to exemplifying production complementarities which can be studied using multilateral matching, the concrete market has engendered a significant literature in industrial organization; see the work of Syverson [35, 36] and Collard-Wexler [12].

[13] We simplify the discussion by omitting other ingredients, such as water and additives.

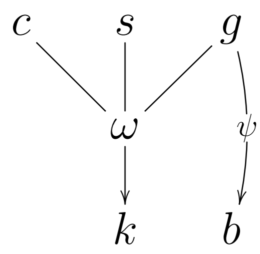

Figure 2: The example economy, reinterpreted as multilateral matching.

where r_χ denotes the number of cubic yards of the good associated with χ that are delivered.

We assume that concrete is produced with increasing marginal cost by k, and the demand of b is bounded above. We also make the simplifying assumption that concrete production requires cement, gravel, and sand in exact 1:2:2 proportions.[14] This gives rise to valuations of the following form:

$$v^k(r_{\omega_c}, r_{\omega_g}, r_{\omega_s}) = 60 \min\left\{\frac{1}{5}r_{\omega_c}, \frac{2}{5}r_{\omega_g}, \frac{2}{5}r_{\omega_s}\right\} - \frac{7}{100}\left(\min\left\{\frac{1}{5}r_{\omega_c}, \frac{2}{5}r_{\omega_g}, \frac{2}{5}r_{\omega_s}\right\}\right)^2,$$

$$v^b(r_\psi) = 32\min\{r_\psi, 50\}.$$

The concavity of the valuation function of k with respect to the amount of concrete produced arises from the fact that k faces an (assumed) downward-sloping demand curve for concrete.[15]

Given the (fixed) proportionality in concrete production, k will never buy disproportionate amounts of cement, gravel, and sand. Thus, we may study the contracting decision of k from the perspective of total concrete production. We represent this by a single *multilateral venture* ω which denotes the production of one cubic yard of concrete using cement, gravel, and sand, as pictured in Figure 2. With this reparameterization, agents' utilities take the following form:

$$\begin{aligned} v^c(r_\omega) &= -16r_\omega, \\ v^g(r_\omega, r_\psi) &= -10r_\omega - 25r_\psi, \\ v^s(r_\omega) &= -2r_\omega - \frac{1}{100}(r_\omega)^2, \\ v^k(r_\omega) &= 60r_\omega - \frac{7}{100}(r_\omega)^2, \\ v^b(r_\psi) &= 32\min\{r_\psi, 50\}. \end{aligned}$$

Since relationships are multilateral, the transfer prices corresponding to a relationship must define payments among all parties to the venture (instead of a single transfer from buyer to seller); hence the transfer prices associated with the venture ω are represented by a vector p_ω such that $p^k_\omega + p^c_\omega + p^g_\omega + p^s_\omega = 0$. Similarly, $p^b_\psi + p^g_\psi = 0$. Agents' utilities are assumed

[14]The assumption of exact proportionality is not necessary but simplifies the exposition.

[15]Alternatively, the same functional form could arise from increasing marginal costs of production.

to be quasilinear in transfers. Given this formulation of prices and utilities, our definition of *competitive equilibrium* is natural: A competitive equilibrium consists of an allocation $r = (r_\omega, r_\psi)$ and a price matrix p such that r is utility-maximizing for every agent given p.

We now construct a competitive equilibrium of our economy; generalizations of this construction show that a competitive equilibrium exists for arbitrary concave valuations (Theorem 3). Our adaptatation of the first welfare theorem to this environment (Theorem 1) shows that all competitive equilibria are efficient in our model, and so we begin by identifying the efficient allocation. Aggregate welfare is given by

$$v^c(r_\omega) + v^g(r_\omega, r_\psi) + v^s(r_\omega) + v^k(r_\omega) + v^b(r_\psi);$$

this is maximized at $(\hat{r}_\omega, \hat{r}_\psi) = (200, 50)$. We now construct a price matrix to support this allocation in competitive equilibrium, demonstrating the second welfare theorem in our environment (Theorem 2). Competitive equilibrium pricing must render $\hat{r}$ individually optimal for each agent; hence we set transfer prices associated with the multilateral venture ω equal to the marginal utility of each agent for an additional unit of production at the efficient allocation; a simple computation shows that $(p^k_\omega, p^c_\omega, p^g_\omega, p^s_\omega) = (32, -16, -6, -10)$. These prices are guaranteed to sum to 0 by the fact that $\hat{r}$ is efficient, from whence it follows that the social marginal utility of adjusting r_ω must vanish. Similarly, we have that $(p^b_\psi, p^g_\psi) = (25, -25)$.[16,17] Note that $\hat{r}$ and p together comprise the unique competitive equilibrium in this model.[18]

Furthermore, the competitive equilibrium above is *stable* in the matching-theoretic sense: No firm desires to unilaterally drop any venture $\chi \in \{\psi, \omega\}$ and associated transfer payments, and no set of firms wishes to renegotiate venture participation levels and transfers. This fact can be shown directly by computation, or as a special case of our Theorem 7.

3. Model

In this section we introduce our general model of multilateral matching. As we demonstrate in Section 6, a large class of economies with production complementarities may be embedded into the multilateral matching framework; this class includes the economy discussed in the previous section.

There is a finite set I of *agents*, and a finite set Ω of *ventures*. Each venture $\omega \in \Omega$ is associated with a set of at least two agents $a(\omega) \subseteq I$; there may be several ventures associated with the same set of agents.[19] For a set of ventures $\Psi \subseteq \Omega$, we denote by $a(\Psi) \equiv \cup_{\psi \in \Psi} a(\psi)$ the set of agents associated with ventures in Ψ. We denote by $\Psi_i \equiv \{\psi \in \Psi : i \in a(\psi)\}$ the

[16] Since the valuation function of b is not differentiable, subgradient calculations are needed in the computation of p_ψ; for details, see the proof of Theorem 2.

[17] This price matrix corresponds to prices of 80, 15, and 25 per cubic yard for cement, gravel, and sand, respectively.

[18] In our general model, the competitive equilibrium is always unique when all valuation functions are continuously differentiable and strictly concave.

[19] Mathematically, the set of agents and the set of ventures together define a multi-hypergraph, where each agent is a node of the graph and each venture is a hyperedge. A hyperedge generalizes the notion of an edge to allow for an arbitrary number of endpoints, instead of just two.

set of ventures in Ψ associated with agent i.

A venture may represent production (of a good such as concrete, as in Section 2), a joint research program, or any other multi-agent endeavor for which participation is continuously adjustable. The possibility of multiple ventures between a given set of agents allows us to encode production processes that do not require fixed input proportions.

We denote by $r_\omega \in [0, r_\omega^{\max}]$ the chosen level of *participation* in venture $\omega \in \Omega$ by the agents in $a(\omega)$; for instance, as in our example in Section 2, if the venture ω is between a supplier of cement, a supplier of gravel, a supplier of sand, and a producer of concrete, r_ω may paramaterize the number of cubic yards of concrete produced. As the notation suggests, we assume that participation in each venture $\omega \in \Omega$ is bounded by some finite bound $r_\omega^{\max} \in \mathbb{R}_{\geq 0}$.

Each agent $i \in I$ has a continuous valuation function $v^i(r)$ over ventures, where the vector $r \equiv (r_\omega)_{\omega \in \Omega}$ is an *allocation* which indicates the investment in each venture $\omega \in \Omega$. Many of our results rely on the assumption that the valuation functions v^i are *concave* in venture participation. This assumption is natural when firms face capacity constraints or when their production technologies exhibit decreasing returns to scale.[20] We assume that v^i is unaffected by ventures to which i is not a party, i.e., $v^i(r_\omega, r_{\Omega \smallsetminus \{\omega\}}) = v^i(\tilde{r}_\omega, r_{\Omega \smallsetminus \{\omega\}})$ for all ω such that $i \notin a(\omega)$.[21]

As illustrated in Section 2, the definitions presented—multilateral ventures and valuation functions—allow us to model production processes with fixed proportions. In fact, these definitions are quite flexible. In addition to proportional production, they can be used to model complementarities between ventures, such as in the case of a Cobb-Douglas production function: To see this, let $I = \{i, j, k\}$ and $\Omega = \{\psi, \omega\}$, with $a(\psi) = \{i, j\}$ and $a(\omega) = \{i, k\}$. If $v^i(r) = (r_\psi)^{\mathfrak{a}}(r_\omega)^{\mathfrak{b}}$ where $\mathfrak{a}, \mathfrak{b} \in [0, 1]$ and $\mathfrak{a} + \mathfrak{b} \leq 1$, then the production technology used by agent i has Cobb-Douglas form.

A venture $\omega \in \Omega$ only represents the nonpecuniary aspects of a transaction between the members of $a(\omega)$. The purely financial aspects of venture ω are represented by a vector p_ω, where p_ω^i is the transfer *price* per unit of the venture that agent i pays in order for the venture ω to transact; this transfer price may be negative if i receives compensation from the other agents in the venture. For any agent $j \notin a(\omega)$, we use the convention that $p_\omega^j \equiv 0$. Furthermore, a venture does not create or use the numeraire; hence $\sum_{i \in I} p_\omega^i = 0$ for all $\omega \in \Omega$.[22] We denote by $p \equiv (p_\omega^i)_{i \in I, \omega \in \Omega}$ the matrix for which p_ω^i is the per-unit transfer from agent i when he engages in venture ω.

An allocation r along with a price matrix p together define an *arrangement* $[r; p]$.[23] The *utilty function* $u^i([r; p])$ of an agent i is quasilinear over ventures and transfer prices, hence

[20] Unfortunately, the concavity of agents' valuations may depend upon the specification of the venture set Ω. The issue of how contractual language interacts with agents' preferences arises throughout matching theory (see Hatfield and Kominers [21]).

[21] We relax this assumption in Section 8 in order to consider contracting externalities.

[22] This assumption is without loss of generality: if a venture ω required a certain amount of numeraire t per unit, it would be equivalent to a venture that did not require the numeraire but imposed an additional utility cost of t on one agent in $a(\omega)$.

[23] We use the term "arrangement" instead of "outcome", as we later we use the term "outcome" to specify a set of contracts.

it can be expressed in the form

$$u^i([r;p]) \equiv v^i(r) - p^i \cdot r.$$

Given prices p, we define the *demand correspondence* $D^i(p)$ for agent i as

$$D^i(p) \equiv \underset{0 \le r \le r^{\max}}{\arg\max}\{u^i([r;p]\}).$$

Since any two allocations which differ only on ventures to which i is not a party provide the same payoff to i, $u^i([r;p])$ does not depend on the size of r_ω for any $\omega \in \Omega \smallsetminus \Omega_i$. Hence, the demand correspondence $D^i(p)$ has the feature that if $(r_{\Omega_i}, r_{\Omega \smallsetminus \Omega_i}) \in D^i(p)$, then $(r_{\Omega_i}, \tilde{r}_{\Omega \smallsetminus \Omega_i}) \in D^i(p)$ for all $\tilde{r}_{\Omega \smallsetminus \Omega_i}$ such that $0 \le \tilde{r}_{\Omega \smallsetminus \Omega_i} \le r^{\max}_{\Omega \smallsetminus \Omega_i}$. We adopt this somewhat unintuitive convention—which typically makes $D^i(p)$ very large—so that we may define the natural demand correspondence for the entire economy as

$$D(p) \equiv \bigcap_{i \in I} D^i(p),$$

which exactly characterizes the levels of investment in each of the (joint) ventures at which all agents' demands are satisfied given prices p.

A *contract* x is comprised of a venture $\omega \in \Omega$, a size of that venture $r_\omega \in [0, r^{\max}_\omega]$, and a transfer vector $s_\omega \in \mathbb{R}^I$ (where we set $s^j_\omega = 0$ for all $j \notin a(\omega)$, maintaining the convention that agents do not receive transfers for ventures to which they are not associated). We study contracts which specify transfers s_ω (instead of per-unit prices p_ω) in order to maintain consistency with the previous literature (e.g., Hatfield et al. [23]); transfers are related to per-unit prices by the formula $s_\omega = r_\omega p_\omega$.

The set of all contracts is

$$X \equiv \left\{ (\omega, r_\omega, s_\omega) \in \Omega \times \mathbb{R}_{\ge 0} \times \mathbb{R}^I : r_\omega \le r^{\max}_\omega,\ s^i_\omega = 0 \text{ for } i \notin a(\omega), \text{ and } \sum_{i \in I} s^i_\omega = 0 \right\}.$$

For $x = (\omega, r_\omega, s_\omega) \in X$, we let $\tau(x) \equiv \omega$; for $Y \subseteq X$ we let $\tau(Y) \equiv \cup_{y \in Y}\{\tau(y)\}$. Analogously to the notation for ventures, for a contract $x \in X$ we let $a(x) \equiv a(\tau(x))$ and for $Y \subseteq X$ we let $a(Y) \equiv a(\tau(Y))$. Similarly, $Y_i \equiv \{y \in Y : i \in a(y)\}$. We define $\kappa([r;p])$ to be the set of contracts that implement the arrangement $[r;p]$, i.e.,

$$\kappa([r;p]) \equiv \{(\omega, \tilde{r}_\omega, \tilde{s}_\omega) \in X : \tilde{r}_\omega = r_\omega > 0 \text{ and } \tilde{s}_\omega = r_\omega p_\omega\}.$$

A set of contracts $Y \subseteq X$ is an *outcome* if it describes a well-defined participation and pricing plan, i.e., if for any $(\omega, r_\omega, s_\omega), (\bar{\omega}, \tilde{r}_{\bar{\omega}}, \tilde{s}_{\bar{\omega}}) \in Y$ such that $(\omega, r_\omega, s_\omega) \neq (\bar{\omega}, \tilde{r}_{\bar{\omega}}, \tilde{s}_{\bar{\omega}})$, we have that $\omega \neq \bar{\omega}$.[24] For any arrangement $[r;p]$, the set of contracts $\kappa([r;p])$ is an outcome.

[24]Without loss of generality, we also impose the requirement that outcomes not include contracts of the form $(\omega, 0, s_\omega)$.

For a given outcome Y, we define $\rho(Y)$ as

$$\rho_\omega(Y) \equiv \begin{cases} r_\omega & (\omega, r_\omega, s_\omega) \in Y \\ 0 & \text{otherwise;} \end{cases}$$

that is, $\rho(Y)$ denotes the associated allocation vector of venture participations. Similarly, we let $\pi(Y)$, where

$$\pi^j_\omega(Y) \equiv \begin{cases} \frac{s^j_\omega}{r_\omega} & (\omega, r_\omega, s_\omega) \in Y \\ 0 & \text{otherwise,} \end{cases}$$

denote the matrix of per-unit transfer prices associated to Y. The *utility* from an outcome Y for agent i is then given by

$$u^i(Y) \equiv v^i(\rho(Y)) - \pi^i(Y) \cdot \rho(Y).$$

The *choice correspondence* of agent i is given by

$$C^i(Y) \equiv \underset{Z \subseteq Y_i; Z \text{ is an outcome}}{\arg\max} \{u^i(Z)\}.$$

4. Competitive Equilibria

We first introduce the *competitive equilibrium* solution concept.

Definition. A *competitive equilibrium* is an arrangement $[r; p]$ such that $r \in D(p)$.

The statement that the arrangement $[r; p]$ is a competitive equilibrium incorporates both individual optimality and market clearing. Individual optimality holds in competitive equilibrium, as each agent i demands the allocation r given the prices p. Furthermore, markets clear in competitive equilibrium, as if an agent $i \in a(\omega)$ demands r_ω at competitive equilibrium prices p, each other agent $j \in a(\omega)$ demands r_ω at those prices.

4.1. Welfare Theorems for Multilateral Matching

In our setting, we obtain results on the relationship between efficient allocations and competitive equilibria that are analogous to the first and second welfare theorems of general equilibrium theory. However, because our setting allows for arbitrarily large transfers of the numeraire, the standard Pareto optimality condition is replaced by (global) efficiency.

An allocation $\hat{r}$ is *efficient* if

$$\hat{r} \in \underset{0 \le r \le r^{\max}}{\arg\max} \sum_{i \in I} v^i(r),$$

i.e., if it maximizes social surplus. Our "First Welfare Theorem" indicates that any competitive equilibrium allocation is efficient.

Theorem 1. *For any competitive equilibrium $[r; p]$, the allocation r is efficient.*

The proof of Theorem 1 uses standard techniques: For any competitive equilibrium $[r; p]$, suppose that some other allocation $\hat{r}$ delivers strictly greater social surplus than r does. Then, since $\sum_{i \in I} p^i_\omega = 0$ for all $\omega \in \Omega$,

$$\sum_{i \in I}(v^i(r) - p^i \cdot r) = \sum_{i \in I} v^i(r) < \sum_{i \in I} v^i(\hat{r}) = \sum_{i \in I}(v^i(\hat{r}) - p^i \cdot \hat{r}). \tag{1}$$

However, the inequality (1) can only hold if there exists an agent j such that

$$v^j(r) - p^j \cdot r < v^j(\hat{r}) - p^j \cdot \hat{r}.$$

But then $r \notin D^j(p)$.

Our "Second Welfare Theorem" gives a partial converse to Theorem 1.

Theorem 2. *Suppose that agents' valuation functions are concave. Then, for any efficient allocation r, there exist prices p such that $[r; p]$ is a competitive equilibrium.*

While the result of Theorem 2 is familiar, the proof, unlike in general equilibrium settings, relies on arguments from differential algebra. The logic is especially transparent in the case that agents' valuation functions are differentiable: In this case, for an efficient allocation r, let $p^i_\omega \equiv \frac{\partial}{\partial r_\omega} v^i(r)$. It follows from the linearity of the differential operator and the fact that r is globally optimal that, for all $\omega \in \Omega$,

$$\sum_{i \in I} p^i_\omega = \sum_{i \in I} \frac{\partial}{\partial r_\omega} v^i(r) = \frac{\partial}{\partial r_\omega} \sum_{i \in I} v^i(r) = 0;$$

hence, p is a valid price matrix. Furthermore, as each v^i is concave, by the construction of p we have that

$$r \in D^i(p)$$

for each $i \in I$. It then follows immediately that $[r; p]$ is a competitive equilibrium.

4.2. Existence of Competitive Equilibria

An immediate consequence of Theorem 2 is that a competitive equilibrium exists in our setting whenever agents' valuation functions are concave.

Theorem 3. *Suppose that agents' valuation functions are concave. Then there exists a competitive equilibrium. If the agents' valuation functions are strictly concave and continuously differentiable, then there exists a unique competitive equilibrium.*

Concavity of agents' valuation functions and the boundedness of the allocation space imply the existence of an efficient allocation $\hat{r}$. Theorem 2 then shows that there exist prices p such that $[\hat{r}; p]$ is a competitive equilibrium. Note that, as preferences are quasilinear in the numeraire, this argument does not require the fixed-point methods used in general equilibrium theory. In fact, the proofs of Theorems 2 and 3 imply a simple algorithm based on Newton's method for computing competitive equilibria in our setting.[25]

[25] As we show in Appendix C, 3 can also be proven by appealing to Kakutani's fixed point theorem, using

Our next result shows that the conditions of Theorem 3 are tight—the domain of concave valuations is the maximal domain for which competitive equilibria are guaranteed to exist.

Theorem 4. *Suppose that the valuation function v^i of some agent i is not concave. Then there exist concave valuation functions for the other agents such that no competitive equilibrium exists.*

To demonstrate the intuition behind this result, consider the case where $I = \{i, j\}$, $\Omega = \{\omega\}$, $a(\omega) = \{i, j\}$, and $r_\omega^{\max} = 2$. Let

$$v^i(r_\omega) = (r_\omega)^2.$$

In this case, $v^i(r_\omega)$ is not concave at $r_\omega = 1$; in fact, $v^i(r_\omega)$ is globally convex. Let

$$v^j(r_\omega) = \begin{cases} 2014 r_\omega & r_\omega \leq 1 \\ 2014(2 - r_\omega) & 1 \leq r_\omega \leq r_\omega^{\max}, \end{cases}$$

which is globally concave. It is clear that the efficient allocation has $r_\omega = 1$. Hence, any competitive equilibrium must be of the form $[(1); (p_\omega)]$. However, for any price p_ω^i, we have that

$$D^i(p) \subseteq \{(0), (2)\},$$

as v^i is globally convex. Hence, no competitive equilibrium exists.

The intuition of the preceding example generalizes to prove Theorem 4: If there is a point at which the valuation function of agent i is not concave, then we construct concave valuation functions for the other agents so that the efficient allocation is at that point. Given that the utility function of agent i is quasilinear in the numeraire, there does not exist a price vector such that it is individually optimal for i to demand an allocation at which his valuation function is not concave. Thus, there does not exist a price vector that induces i to demand the allocation that is efficient in the constructed economy. Hence, since by Theorem 1 all competitive equilibria are efficient, no competitive equilibrium exists.

4.3. Comparative Statics

We now prove an intuitive comparative static result: as an individual venture ψ becomes more valuable for the agents in $a(\psi)$, those agents will not choose to participate in ψ less than before.[26]

Theorem 5. *Consider a family of valuation functions $v^i(\cdot; \ell)$ parameterized by ℓ. Suppose that for all $i \in I$, v^i is strictly concave in r for all $\ell \in \mathbb{R}$ and is twice continuously differen-*

an approach similar to the standard techniques of general equilibrium theory. The Kakutani fixed point approach, however, has the disadvantage that it not does not provide an explicit method for computing competitive equilibria.

[26]Note that we can not characterize how participation in any other venture ξ changes as ψ becomes more valuable, as ψ and ξ may act as either complements or substitutes.

tiable in r and ℓ. Suppose additionally that for all $i \in I$, all $\ell \in \mathbb{R}$, and some $\psi \in \Omega$,

$$\frac{\partial^2 v^i(r;\ell)}{\partial r_\psi \partial \ell} \geq 0 \text{ and}$$
$$\frac{\partial^2 v^i(r;\ell)}{\partial r_\omega \partial \ell} = 0 \text{ for all } \omega \in \Omega \text{ such that } \omega \neq \psi.$$

Let $[\hat{r}(\ell); \hat{p}(\ell)]$ be the unique competitive equilibrium in the economy (for the parameter ℓ) implied by Theorem 3. Then,

$$\frac{\partial \hat{r}_\psi(\ell)}{\partial \ell} \geq 0.$$

Note that under the conditions of Theorem 5, the efficient allocation $\hat{r}(\ell)$ is unique. As venture ψ becomes more profitable for the agents in $a(\psi)$, the conditions on r_ψ in the global optimization problem slacken. The implicit function theorem then shows that the efficient level of participation in ψ must increase. Since the competitive equilibrium allocation is efficient, the value of $\hat{r}_\psi$ must therefore also increase.

5. Cooperative Solution Concepts

5.1. Definitions

We now introduce the standard notion of *stability* from the matching literature.[27]

Definition. An outcome A is *stable* if it is

1. *individually rational*: for all $i \in I$, $A_i \in C^i(A)$;
2. *unblocked*: there does not exist a nonempty $Z \subseteq X \smallsetminus A$ such that for all $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$.

Individual rationality of A requires that no agent i prefer to drop some of the contracts in A_i. Unblockedness of A requires that there does not exist a new set of contracts Z such that all the agents in $a(Z)$ would strictly prefer to sign all the contracts in Z (and possibly drop some of their existing contracts in A) rather than only sign some (or none) of them.[28]

Closely related to stability is the standard solution concept of cooperative game theory: the *core*.

Definition. An outcome A is in the *core* if it is *core unblocked*: there does not exist a nonempty $Z \subseteq X$ such that $u^i(Z) > u^i(A)$ for all $i \in a(Z)$.

The definition of the core differs from that of stability in two ways. First, core unblockedness requires that all of the blocking agents (i.e., those agents in $a(Z)$) drop all of

[27] Note that unlike in classical matching theory, we must consider the possibility of indifference between two sets of contracts; hence, we use the definition of Hatfield et al. [23].

[28] Note that our stability concept allows agents associated with a blocking set to disagree on whether a contract in the original allocation is maintained while deviating. As can easily be checked, ruling out such blocks does not weaken our results.

their prior contracts (i.e., those contracts in $A \smallsetminus Z$); this is a more stringent restriction than that of stability, which allows agents in $a(Z)$ to retain previous relationships. Second, core unblockedness does not require that the deviation be individually optimal for each deviating agent (i.e., it need not be the case that for all $i \in a(Z)$, $Z_i \subseteq Y^i$ for each $Y^i \in C^i(Z \cup A)$); rather, it requires only that the weaker condition that each blocking agent is made better off by the deviation (i.e., $u_i(Z) > u_i(A)$).

Finally, we introduce *strong group stability*, first proposed by Hatfield et al. [23], which is a stronger solution concept than both the core and stability.

Definition. An outcome A is *strongly group stable* if it is:

1. individually rational;
2. *strongly unblocked*: There does not exist a nonempty set $Z \subseteq X \smallsetminus A'$ such that for all $i \in a(Z)$ there exists a $Y^i \subseteq Z \cup A$ such that $Z_i \subseteq Y^i$ and $u^i(Y^i) > u^i(A)$.

Strong group stability is more restrictive than the core—unlike coalitional unblockedness, strong unblockedness does not require agents to drop previous relationships. Additionally, strong group stability is more restrictive than stability, as strong unblockedness does not require that the deviation set Z be individually optimal for each deviating agent (i.e., it need not be the case that for all $i \in a(Z)$, $Z_i \subseteq Y^i$ for each $Y^i \in C^i(Z \cup A)$), but only that there exists a utility-increasing individual-specific deviation for each agent (i.e., a $Y^i \supseteq Z_i$ such that $u^i(Y^i) > u^i(A)$).[29]

5.2. The Relationship between Cooperative Solution Concepts

The following result is immediate from the definitions.

Theorem 6. *If an outcome Y is strongly group stable, then Y is stable and in the core. Furthermore, all core allocations are efficient.*

In general, there are no relationships between the cooperative solution concepts beyond those in Theorem 6 without additional assumptions on the valuation functions. To see this, suppose $\Omega = \{\psi, \omega\}$, $a(\omega) = a(\psi) = I = \{i, j\}$, and $r_\psi^{\max} = r_\omega^{\max} = 1$. Let the valuation functions of the two agents be given by

$$v^i(r) = 7 \min\{r_\psi, r_\omega\}$$
$$v^j(r) = -6 \min\{r_\psi, r_\omega\}.$$

The unique efficient allocation is $r = (1, 1)$. It follows that the core is given by

$$\{\{(\psi, r_\psi, (s_\psi^i, s_\psi^j)), (\omega, r_\omega, (s_\omega^i, s_\omega^j))\} \subseteq X : r_\psi = r_\omega = 1, 6 \le s_\psi^i + s_\omega^i \le 7\}.$$

[29]This concept is called strong group stability as it is stronger than the existing concepts of *strong stability* and *group stability*. Strong stability (introduced by Hatfield and Kominers [21]) also requires that each Z_i be individually rational. Group stability (introduced by Roth and Sotomayor [34] and extended to the setting of many-to-many matching by Konishi and Ünver [28]) requires that if $y \in Y^i$ for some $i \in a(Y)$, then $y \in Y^j$ for all $j \in a(y)$, i.e., that the deviating agents agreed on which contracts from the original allocation would be kept after deviation. Hatfield et al. [23] provide a further discussion.

However, no core outcome is stable. Suppose, without loss of generality, that $s^i_\psi \geq s^i_\omega$. Since $s^i_\psi + s^i_\omega \leq 7$ we have that $s^i_\omega \leq \frac{7}{2}$. Then agent j will choose to drop the contract $(\omega, 1, (s^i_\omega, s^j_\omega))$ as it costs him 6 (due to the cost of production) but gains him at most $\frac{7}{2}$ in transfer.

However, the outcome $\varnothing$ is stable. Consider a blocking set of the form $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi))\}$ or $\{(\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$; since no agent gains benefits or incurs costs from such a set of contracts, each agent is indifferent between this set of contracts and $\varnothing$. For a blocking set of the form $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi)), (\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$ we must have that $s^i_\psi + s^i_\omega > -7 \min\{r_\psi, r_\omega\}$ as i must choose both contracts; this implies that $s^j_\psi + s^j_\omega < 7 \min\{r_\psi, r_\omega\}$. Suppose without loss of generality that $s^j_\psi \geq s^j_\omega$; then $s^j_\omega < \frac{7}{2} \min\{r_\psi, r_\omega\} < 6 \min\{r_\psi, r_\omega\}$. Hence j strictly prefers $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi))\}$ to $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi)), (\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$ and so $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi)), (\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$ is not a blocking set.

The preceding example illustrates that, in general, there is no logical relationship between stable and core outcomes.[30] Appendix D gives an example of an outcome that is both stable and core, but is not strongly group stable.

5.3. The Relationship between Stable Outcomes and Competitive Equilibria

We now show that every competitive equilibrium is associated with a stable outcome.

Theorem 7. *Suppose that $[r; p]$ is a competitive equilibrium. Then, $\kappa([r; p])$ is (strongly group) stable and in the core.*

The proof of Theorem 7 is similar to, but more technical than, the proof of Theorem 1 sketched in Section 4.2. If $\kappa([r; p])$ is not individually rational, then $\kappa([r; p])_i \notin C^i(\kappa([r; p]))$, which implies that $r \notin D^i(p)$, so $[r; p]$ is not a competitive equilibrium. If $\kappa([r; p])$ is not strongly unblocked, then there is a set Z such that for all $i \in a(Z)$, there exists $Y^i \supseteq Z_i$ such that $u^i(Y^i) > u^i(\kappa([r; p]))$. Summing over individuals, and using the fact that $\pi^i(\kappa([r; p])) \cdot \rho(\kappa([r; p])) = p^i \cdot \rho(\kappa([r; p]))$, we obtain

$$\sum_{i \in a(Z)} v^i(\rho(Y^i)) - \pi^i(Y^i) \cdot \rho(Y^i) > \sum_{i \in a(Z)} v^i(\rho(\kappa([r; p]))) - p^i \cdot \rho(\kappa([r; p])).$$

Since transfers among agents in $a(Z)$ sum to 0, we have that $\sum_{i \in a(Z)} \pi^i_\omega(Y^i) = 0 = \sum_{i \in a(Z)} p^i_\omega$ for each $\omega \in \tau(Z)$. Hence,

$$\sum_{i \in a(Z)} v^i(\rho(Y^i)) - p^i \cdot \rho(Y^i) > \sum_{i \in a(Z)} v^i(\rho(\kappa([r; p]))) - p^i \cdot \rho(\kappa([r; p])).$$

But then there must exist $j \in a(Z)$ such that $u^j(\kappa([\rho(Y^j); p])) > u^j(\kappa([r; p]))$, and hence

$$r = \rho(\kappa([r; p])) \notin D^j(p),$$

implying that $[r; p]$ is not a competitive equilibrium.

[30] Moreover, since no outcome in this example is both stable and in the core, no strongly group stable outcome exists.

An immediate corollary of Theorems 3 and 7 is the existence of strongly group stable outcomes for concave valuation functions.

Corollary 1. *Suppose that agents' valuation functions are concave. Then a (strongly group) stable outcome exists.*

The converse of Theorem 7 is not true: not all (strongly group) stable outcomes correspond to competitive equilibria. Consider the case where there are two agents i and j, and two ventures ψ and ω. Suppose that

$$v^i(r) = -4 \max\{r_\psi, r_\omega\}$$
$$v^j(r) = 3 \max\{r_\psi, r_\omega\}.$$

Then $\varnothing$ is a (strongly group) stable outcome; however, no competitive equilibrium exists. As Theorem 1 shows, every competitive equilibrium is efficient, hence any competitive equilibrium must be of the form $[(0,0); p]$. For any price matrix p, we must have that

$$\min\{p^j_\psi, p^j_\omega\} \geq 3$$

as otherwise agent j will demand positive amounts of ψ or ω. This implies that

$$p^i_\psi + p^i_\omega \leq -6,$$

as $p^j_\psi = -p^i_\psi$ and $p^j_\omega = -p^i_\omega$. Then agent i will demand positive amounts of both ψ and ω; hence no prices will support $r = (0,0)$ as a competitive equilibrium.

The example above relies on the fact that the valuation function of j is not concave. Our next two results show that the lack of concavity is essential for the example: when all agents have concave valuation functions, every stable outcome corresponds to an efficient allocation, and hence induces a competitive equilibrium.

Theorem 8. *Suppose that agents' valuation functions are concave. Then, for any stable outcome A, the allocation $\rho(A)$ is efficient.*

When all agents' valuation functions are concave, at any outcome A corresponding to an inefficient allocation $r = \rho(A)$, either A is not individually rational or there exists a venture ψ such that the total welfare of the agents in $a(\psi)$ can be increased by adjusting r_ψ to some other value $\tilde{r}_\psi$. The agents in $a(\psi)$ can then choose transfers $\tilde{s}_\psi$ so as to share the surplus from adjusting r_ψ to $\tilde{r}_\psi$. By construction, then, it follows that $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\}$ blocks A. Thus, if A is stable, then $\rho(A)$ is efficient.

Combining Theorems 2 and 8, we immediately obtain the following corollary.

Corollary 2. *Suppose that agents' valuation functions are concave. Then, for any stable outcome A, there exists a price matrix p such that the arrangement $[\rho(A); p]$ is a competitive equilibrium.*

An analogous result holds for core outcomes, since those outcomes are efficient by Theorem 6.

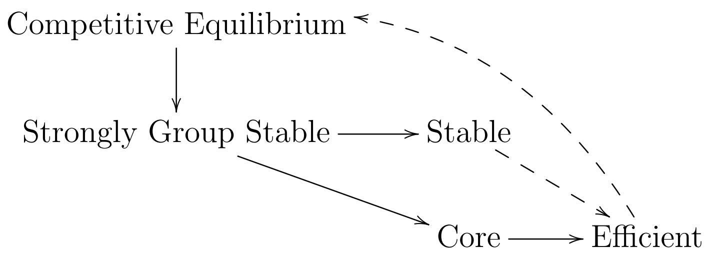

Figure 3: The Relationships Between the Solution Concepts.

Corollary 3. *Suppose that agents' valuation functions are concave. Then, for any core outcome A, there exists a price matrix p such that the arrangement $[\rho(A); p]$ is a competitive equilibrium.*

Note that Corollaries 2 and 3 imply that the underlying allocations of stable and core outcomes can be supported in competitive equilibrium, but do not imply any relationship between the supporting prices and the transfers associated with the original outcomes. These results are analagous to results in general equilibrium theory, where the set of utilities induced by core outcomes is also, in general, larger than the set of utilities induced by competitive equilibria.

We summarize the relationship between the various solution concepts in Figure 3. Solid lines indicate relationships which hold in general; dashed lines represent relationships that hold in the presence of concave valuations.

6. Application: Production Economies with Complementary Inputs

We now demonstrate how our model can be applied to economies where production requires complementary inputs. In Appendix E, we show how the example presented in Section 2 can be described as such an economy.

Consider an economy with a set of agents I in which each agent $i \in I$ has an initial endowment $e^i_{\mathsf{g}} \geq 0$ of each good g; the set of all goods is denoted G. Available to the agents are a set of production processes Ω. Each $\omega \in \Omega$ is represented by a matrix in $\mathbb{R}^{I \times \mathsf{G}}$, where the value ω^i_{g} indicates that i obtains ω^i_{g} units of good g per unit of process ω executed. If $\omega^i_{\mathsf{g}} > 0$, then good g is an output of the process for agent i; if $\omega^i_{\mathsf{g}} < 0$, then good g is an input of the process for agent i. Note that these processes need not result in the creation or destruction of goods: For example, the process

$$\psi^i_{\mathsf{g}} = \begin{cases} -1 & i = j \text{ and } \mathsf{g} = \mathsf{x} \\ 1 & i = k \text{ and } \mathsf{g} = \mathsf{x} \\ 0 & \text{otherwise} \end{cases}$$

denotes the transfer of one unit of good x from agent j to agent k. In contrast, a linear production process of the form

$$\chi^i_{\mathsf{g}} = \begin{cases} -1 & i = j \text{ and } \mathsf{g} = \mathsf{x} \\ -2 & i = k \text{ and } \mathsf{g} = \mathsf{y} \\ 1 & i = h \text{ and } \mathsf{g} = \mathsf{z} \\ 0 & \text{otherwise} \end{cases}$$

denotes the production of one unit of good z by agent h using one unit of good x from agent j and two units of good y from agent k.[31] We denote by $r_\omega \leq r^{\max}_\omega$ the quantity of engagement in process ω; for instance, $r_\psi = 2$ (where ψ is as defined above) indicates the transfer of two units of good x from i to j.

The final consumption of agent i is given by a vector $\mathcal{c}^i$, where

$$\mathcal{c}^i_{\mathsf{g}}(r) = \mathcal{e}^i_{\mathsf{g}} + \sum_{\omega \in \Omega} r_\omega \omega^i_{\mathsf{g}}.$$

Each agent $i \in I$ has a continuous valuation function over consumption, denoted $\dot{v}^i(\mathcal{c}^i)$. Note that since the production processes we have specified are linear, all production costs are implicitly embedded into agents' valuations.[32] Thus, the valuations $\dot{v}^i$ are concave when production exhibits nonincreasing returns to scale and scope, and agents receive diminishing marginal utility from final consumption.

The economy just described may be reinterpreted as a multilateral matching economy with agent set I, venture set Ω, and valuation functions

$$v^i(r) = \dot{v}^i(\mathcal{c}^i(r)).$$

It is clear that v^i is concave if $\dot{v}^i$ is, as then, for all $\mathfrak{a} \in [0, 1]$,

$$\begin{aligned} \mathfrak{a} v^i(r) + (1 - \mathfrak{a}) v^i(\tilde{r}) &= \mathfrak{a} \dot{v}^i(\mathcal{c}^i(r)) + (1 - \mathfrak{a}) \dot{v}^i(\mathcal{c}^i(\tilde{r})) \\ &\geq \dot{v}^i(\mathfrak{a} \mathcal{c}^i(r) + (1 - \mathfrak{a}) \mathcal{c}^i(\tilde{r})) \\ &= \dot{v}^i(\mathcal{c}^i(\mathfrak{a} r + (1 - \mathfrak{a}) \tilde{r})) \\ &= v^i(\mathfrak{a} r + (1 - \mathfrak{a}) \tilde{r}), \end{aligned}$$

[31] Note that while each process specifies the exact amount of each input good to be provided by each agent, it is possible that two processes χ and ω might have the same net inputs and outputs (i.e., $\sum_{i \in I} \omega^i_{\mathsf{g}} = \sum_{i \in I} \chi^i_{\mathsf{g}}$ for all $\mathsf{g} \in \mathsf{G}$) but differ in the idenities of the providers of the inputs and recipients of the outputs. Similarly, mulitple processes may produce the same quantity of good g for i, but use different mixtures of inputs.

[32] For illustration, consider an economy with a single good g, which can be produced by the agent $i \in I$ via process ω (using goods from other agents). When using process ω, agent i incurs a convex cost of production $\mathfrak{c}(\mathcal{c}^i_{\mathsf{g}})$. After production, i receives linear utility from consuming good g. The valuation $\dot{v}^i$ then takes the form $\dot{v}^i(\mathcal{c}^i) = \mathcal{c}^i_{\mathsf{g}} - \mathfrak{c}(\mathcal{c}^i_{\mathsf{g}})$.

Note that this convention also allows us to consider the case where an agent can produce the same product at two different factories (with appropriate inputs); we specify the set of goods G to include a good for the product of each factory, so that we may model utility from total consumption alongside convex costs of production at each factory.

where the inequality follows from the concavity of $\dot{v}^i$ and the subsequent equality follows from the linearity of $\underline{c}^i$. Thus agents' valuations are concave whenever their underlying preferences over goods are concave.

The preceding discussion shows that multilateral matching encompasses a large class of economies with production complementarites. Unlike general equilbrium theory, the multilateral matching framework allows us to model economies with agent-specific production, i.e., production that relies on technologies available only to certain agents.[33]

The class of economies with production complementarities includes standard examples from manufacturing, such as the assembly of automobiles and computers. Additionally, this class encompasses economies in which production requires many complementary inputs but the value of the output is uncertain; real world examples of such economies include the oil and gas industries.[34] By contrast, the formation of joint research ventures between firms is not adequately modeled in a production economy setting; nonetheless, it is apparent that research venture formation may be modeled using multilateral matching.[35]

7. Application: Competitive Equilibrium in Economies with a Continuum of Agents

In this section, we demonstrate how our results can be used to derive the existence of competitive equilibria (and stable outcomes) in the large market setting of Azevedo et al. [9]. In the Azevedo et al. [9] setting, there is a set of actor types J and a set of goods G; each actor of type j has a utility function that is quasilinear in the numeraire but is otherwise arbitrary with respect to the valuation over goods.[36] In particular, the type of actor j is specified by a vector $w^j \in \mathbb{R}^{\wp(G)}$, where $\wp(\cdot)$ denotes the power set operator and $w^i_{\bar{G}}$ denotes the value of an actor of type j for the bundle of goods $\bar{G} \subseteq G$; for simplicity, we normalize $w^j_{\varnothing} = 0$ for all $j \in J$.

For each good $g \in G$, there exists quantity q^g of good g; since utility is quasilinear in the numeraire, we do not model initial ownership explicitly. We consider the case where there are a finite set of actor types, and denote by m^j the measure of actors of type j. An *allocation* is a measurable map $A : J \to \Delta\wp(G)$, where $\Delta\wp(G)$ is the finite-dimensional simplex over subsets of the set of goods G. An allocation is feasible if $0 \leq A^j_{\bar{G}} \leq 1$ for all $j \in J$ and $\bar{G} \subseteq G$, and

$$\sum_{j \in J} \left(\left(\sum_{\{g\} \subseteq \bar{G} \subseteq G} A^j_{\bar{G}} \right) m^j \right) = q^g$$

for all $g \in G$; that is, the full quantity of each good is allocated to actors.

[33] Agent-specificity may be material, for instance, in economies with intellectual property rights or implicit knowledge gained from learning-by-doing.

[34] To model such economies in our framework, it suffices that the valuation functions of agents with uncertain outcomes incorporate an expectation operator.

[35] As we remarked in Footnote 10, examples such as joint research ventures show the true strength of our framework. While economies with production complementarities may be modeled using only bilateral contracting, economic activities which exhibit externalities across contractual partners require the full generality of multilateral contracting.

[36] In particular, valuations are not assumed to be substitutable in the sense of Gul and Stacchetti [20].

We denote by $t_g \in \mathbb{R}$ the price of good $g \in G$. The *demand* of an actor of type $j \in J$, as a function of the price vector $t \in \mathbb{R}^G$, is given by

$$D^j(t) \equiv \arg\max_{\bar{G} \subseteq G} \left\{ w^j_{\bar{G}} - \sum_{g \in \bar{G}} t_g \right\}.$$

A *competitive equilibrium* in this setting is an allocation–price pair $[A; t]$ such that A is feasible and, if $A^j_{\bar{G}} > 0$, then $\bar{G} \in D^j(t)$.

Azevedo et al. [9] show the existence of competitive equilibria in this setting.

Theorem 9 (Azevedo et al. [9]). *A competitive equilibrium* $[A; t]$ *exists.*

We now use the multilateral matching framework to provide a novel proof of Theorem 9. To do so, we construct a multilateral matching economy by letting the set of agents be $I = J \cup G$ and specifying a venture $\omega^j_{\bar{G}}$ for each bundle of goods $\bar{G} \supsetneq \varnothing$ that can be assigned to agent type j; hence $a(\omega^j_{\bar{G}}) = \{j\} \cup \bar{G}$.

The valuation of an agent $g \in G$ is given by

$$v^g(r) = -\mathfrak{K} \left| \left(\sum_{\omega \in \Omega_g} r_\omega \right) - q^g \right|,$$

where $\mathfrak{K} > 2 \max_{\bar{G} \subseteq G} |w^j_{\bar{G}}|$; this ensures that, in any competitive equilibrium $[r; p]$, exactly the full quantity of each good g is sold through ventures, i.e.,

$$\left(\sum_{\omega \in \Omega_g} r_\omega \right) - q^g = 0. \tag{2}$$

The valuation of an agent $j \in J$ is given by

$$v^j(r) = \begin{cases} \sum_{\omega \in \Omega_j} w^j_{a(\omega) \smallsetminus \{j\}} r_\omega & \sum_{\omega \in \Omega_j} r_\omega \leq m^j \\ \sum_{\omega \in \Omega_j} w^j_{a(\omega) \smallsetminus \{j\}} r_\omega - \mathfrak{L} \max \left\{ 0, \left(\sum_{\omega \in \Omega_j} r_\omega \right) - m^j \right\} & \text{otherwise,} \end{cases}$$

where $\mathfrak{L} > \max_{\bar{G} \subseteq G} \{|w^j_{\bar{G}}|\} + \mathfrak{K}|G|$; this ensures that

$$\sum_{\omega \in \Omega_j} r_\omega \leq m^j, \tag{3}$$

that is, no actor type is assigned a greater measure of bundles than the measure of actors of that type.

It is immediate from Theorem 3 that a competitive equilibrium $[r; p]$ exists in the constructed multilateral matching economy, as the valuation function for each $i \in J \cup G$ is concave.

Unfortunately, under the arrangement $[r; p]$, it may not be the case that each good "sells" for the same price in all bundles, since our framework allows for personalized, bundle-specific prices. However, for each $g \in G$, since agent g is optimizing (i.e., $r \in D^g(p)$) and the marginal returns to g of all ventures in Ω_g are equal:

- The agent g must be indifferent among participating in ventures that actually transact, given the price matrix p, i.e., for each $\psi, \omega \in \Omega_g$ such that $r_\psi > 0$ and $r_\omega > 0$, $p^g_\psi = p^g_\omega$.
- The agent g must weakly prefer participating in any venture $\omega \in \Omega_g$ such that $r_\omega > 0$ to participating in any venture $\psi \in \Omega_g$ such that $r_\psi = 0$, i.e., $p^g_\omega < p^g_\psi$.[37]

Consider the arrangement $[r; \bar{p}]$ where $\bar{p}^g_\omega = \min_{\psi \in \Omega_g}\{p^g_\psi\}$. For each $\omega \in \Omega$, since $|a(\omega) \cap J| = 1$, knowing $\bar{p}^g_\omega$ for each $g \in G$ pins down the price $\bar{p}^j_\omega$ for the unique $j \in (a(\omega) \cap J)$:

$$\bar{p}^j_\omega = -\sum_{g \in (a(\omega) \smallsetminus \{j\})} \bar{p}^g_\omega.$$

Of course, $\bar{p}^i_\omega = 0$ for all $i \notin a(\omega)$.

The preceding observations imply that, for each venture ω such that $r_\omega > 0$, we have that $\bar{p}_\omega = p_\omega$, as for any such ω, and any $g \in (a(\omega) \cap G)$,

$$p^g_\omega = \min_{\psi \in \Omega_g}\{p^g_\psi\} = \bar{p}^g_\omega.$$

The preceding observations also imply that, for each $j \in J$ and venture ω such that $r_\omega = 0$, we have $\bar{p}^j_\omega \geq p^j_\omega$, as for any such ω, and any $g \in (a(\omega) \cap G)$,

$$p^g_\omega \geq \min_{\psi \in \Omega_g}\{p^g_\psi\} = \bar{p}^g_\omega.$$

Hence, the arrangement $[r; \bar{p}]$ is a competitive equilibrium: For each $j \in J$, prices of ventures which j does not demand have weakly increased relative to $[r; p]$, while ventures j does demand have not changed in price. For each $g \in G$, the prices of ventures in which g is participating are still weakly higher than those of the ventures in which he is not participating.

The arrangement $[r; \bar{p}]$ of the multilateral matching economy induces an arrangement $[\bar{A}; \bar{t}]$ of the Azevedo et al. [9] economy, with

$$\bar{A}^j_{\bar{G}} = \frac{r_{\omega^j_{\bar{G}}}}{m^j}$$

for all $\bar{G} \neq \varnothing$,

$$\bar{A}^j_\varnothing = 1 - \frac{\sum_{\omega \in \Omega_j} r_\omega}{m^j},$$

and $\bar{t}_g = -\bar{p}^g_\omega$ (for any $\omega \in \Omega_g$).

Lemma 1. *The allocation price pair $[\bar{A}; \bar{t}]$ is a competitive equilibrium of the Azevedo et al. [9] economy.*

[37] Recall that in the multilateral matching framework, the price p^g_ω denotes the price *paid by* g so the revenue of g is higher when p^g_ω is lower.

Lemma 1, which we prove in Appendix B, immediately implies Theorem 9.

Note that, unlike the argument of Azevedo et al. [9], this method based on the multilateral matching framework avoids the use of Kakutani's fixed point theorem, and in particular only relies on solving a concave maximization problem. Thus, our approach, unlike that of Azevedo et al. [9], can be used to efficiently compute competitive equilibria of the Azevedo et al. [9] economy.

8. Extension: Markets with Externalities

In this section, we incorporate externalites into our model by relaxing the assumption that $v^i(r_\omega, r_{\Omega\smallsetminus\{\omega\}}) = v^i(\tilde{r}_\omega, r_{\omega\smallsetminus\{\omega\}})$ for all $\omega \in \Omega$ such that $i \notin a(\omega)$. For clarity, throughout this section we express the valuation function v^i of agent i as $v^i(r_{\Omega_i}; r_{\Omega\smallsetminus\Omega_i})$, to highlight the fact that i treats participation in ventures to which he is not a party as exogenous. Abusing terminology slightly, we will say that $v^i(r_{\Omega_i}; r_{\Omega\smallsetminus\Omega_i})$ is *concave* if it is concave in the venture participation r_{Ω_i} of agent i for all $r_{\Omega\smallsetminus\Omega_i}$. Note that we allow for arbitrary externalities so long as each $v^i(r)$ is continuous in $r = (r_{\Omega_i}, r_{\Omega\smallsetminus\Omega_i})$.

We must now consider demand functions of the form

$$\bar{D}^i(p; \tilde{r}) \equiv \underset{0\le r\le r^{\max}}{\arg\max}\{v^i(r_{\Omega_i}; \tilde{r}_{\Omega\smallsetminus\Omega_i}) - p^i \cdot r\},$$

where the additional input $\tilde{r}$ highlights the dependence of the demand of agent i on the venture participation of other agents, $\tilde{r}_{\Omega\smallsetminus\Omega_i}$. As in the case without externalities, the demand correspondence $\bar{D}^i(p; \tilde{r})$ has the feature that if $(r_{\Omega_i}, r_{\Omega\smallsetminus\Omega_i}) \in \bar{D}^i(p; \tilde{r})$, then $(r_{\Omega_i}, \check{r}_{\Omega\smallsetminus\Omega_i}) \in \bar{D}^i(p; \tilde{r})$ for all $\check{r}_{\Omega\smallsetminus\Omega_i}$ such that $0 \le \check{r}_{\Omega\smallsetminus\Omega_i} \le r^{\max}_{\Omega\smallsetminus\Omega_i}$; this allows us to define the demand correspondence for the entire economy by

$$\bar{D}(p; \tilde{r}) \equiv \bigcap_{i\in I} \bar{D}^i(p; \tilde{r}).$$

In this context a *competitive equilibrium* is an arrangement $[r; p]$ such that $r \in \bar{D}(p; r)$.

Our next theorem shows that competitive equilibria exist when agents' valuation functions are concave—even in the presence of externalities.

Theorem 10. *Suppose that agents' valuation functions $v^i(r_{\Omega_i}; r_{\Omega\smallsetminus\Omega_i})$ are concave (in r_{Ω_i}). Then a competitive equilibrium exists.*

Unlike the proof of Theorem 3, the proof of Theorem 10 relies on fixed-point methods. In particular, we use Kakutani's fixed point theorem to show that

$$F(\tilde{r}) \equiv \underset{0\le r\le r^{\max}}{\arg\max}\left\{\sum_{i\in I} v^i(r_{\Omega_i}; \tilde{r}_{\Omega\smallsetminus\Omega_i})\right\}$$

has a fixed point $\hat{r}$. Arguments analogous to the the proof of Theorem 2 show that there exist prices that support $\hat{r}$ in competitive equilibrium. Note, however, that competitive equilibria in the presence of externalities are generally not efficient.[38]

While this approach allows us to find competitive equilibria in settings with externalities, the added generality comes at a cost: we must use Kakutani's fixed point theorem rather than the differential (and easily computable) method used to prove Theorem 3.

Stable outcomes correspond to competitive equilibria in the presence of externalities if agents, when considering whether to choose contracts in a blocking set Z, assume that no other contracts will change.[39] If, however, agents are able to accurately predict that contracts in $Z \smallsetminus Z_i$ will transact, then competitive equilibria may not correspond to stable outcomes.[40]

To see the difference between the two stability notions, suppose that $I = \{h, i, j, k\}, \Omega = \{\psi, \omega\}$ where $a(\psi) = \{i, j\}$, $a(\omega) = \{h, k\}$, and $r^{\max} = (1, 1)$. Let

$$v^i(r_\psi; r_\omega) = -r_\psi, \qquad v^h(r_\omega; r_\psi) = -r_\omega,$$
$$v^j(r_\psi; r_\omega) = 3r_\psi r_\omega, \qquad v^k(r_\omega; r_\psi) = 3r_\omega r_\psi.$$

These valuations may be interpreted as indicating that i and h sell raw materials to j and k respectively, and that there is only a market for the product of j if k sells its product and vice versa. In this setting, there are two competitive equilibrium allocations: $(0, 0)$ and $(1, 1)$. For the allocation $(0, 0)$, the only supporting price matrix is 0; each pair ($\{i, j\}$ and $\{h, k\}$) is unwilling to begin production without the other pair doing so as well, so the set $Z = \{(\psi, 1, (2, -1, 0, 0)), (\omega, 1, (0, 0, 2, -1))\}$ blocks $\varnothing$ if and only if every agent expects all of the other agents to choose their contracts in Z.[41]

9. Conclusion

Our work shows that matching theory can incorporate certain forms of complementarity so long as contracts are continuously divisible. In that case, when agents' valuation functions are concave, competitive equilibria exist, correspond to (strongly group) stable outcomes, and yield core outcomes. Analogues of the first and second welfare theorems hold as well. Even in the presence of externalities, competitive equilibria exist so long as agents' valuations are concave. Further work is needed, however, to identify the appropriate notion of stability for matching models with externalities and to characterize the relationship between that stability concept and the concept of competitive equilibrium.

Previous matching models have obtained conclusions similar to ours—existence and correspondence results for competitive equilibria and stable outcomes (in the presence of quasi-linear utility). However, these results have depended crucially on the presence of (full) preference substitutability, which rules out complementarities of the types encoded in our model's multilateral contracts. The key distinction between the prior work and our model is in the structure of the contractual space: whereas previous models have typically allowed agents to contract over discrete participation levels, we require instead that agents be al-

[38] See Appendix F for a simple example.

[39] It is clear that if $[r; p]$ is a competitive equilibrium, then $\kappa([r; p])_i$ is individually rational for all $i \in I$.

[40] This distinction in the stability of competitive equilibrium outcomes is analogous to the distinction between Cournot and consistent conjectures (Bresnahan [11]) equilibria in oligopoly theory.

[41] When writing transfer vectors, we list transfers in the alphabetical order of agents.

lowed to continuously adjust participation. Our work therefore reveals a tradeoff between modeling assumptions: when contract participation levels are discrete, complementarities must be assumed away, while when they are continuous, some complementarities can be incorporated.

Assuming contractual divisibility seems reasonable in a number of industrial settings, such as chemical synthesis, assembly of durable goods, and automobile manufacturing (Fox [17, 18]). It also seems appropriate in the context of online advertising, where billions of impressions are sold. Multilateral matching models allow us to understand the market outcomes in these settings; they may also prove useful for both empirical work and market design applications in settings with complementarities. Moreover, contactual divisibility can be used to model markets with a continuum of agents, as Section 7 demonstrates.

Meanwhile, divisibility may not be a reasonable assumption for markets where each individual product is unique and of a discretely specified size. In those markets, other analytical tools are needed; there, "large market" effects may facilitate analysis (Kojima et al. [27], Ashlagi et al. [7], Azevedo et al. [9], Azevedo and Hatfield [8]).

Appendix A. Relationship with General Equilibrium

In this appendix, we show that there is a natural embedding of the multilateral matching framework into the standard model of general equilibrium when agents' valuations are monotonic with respect to participation in each venture. We also provide an example that shows that this correspondence does not extend naturally to economies where valuation monotonicity fails.

For simplicity, we consider the case where valuation functions are differentiable and strictly concave; we also assume that $r^{\max}$ is large enough that $\hat{r} < r^{\max}$ at any efficient allocation $\hat{r}$.

Definition. We say that valuation function v^i is *monotonic* if for each $\omega \in \Omega$ and $r \in \mathbb{R}^\Omega$ such that $0 \le r \le r^{\max}$, either

$$\frac{\partial v^i(r)}{\partial r_\omega} > 0 \quad \text{or} \quad \frac{\partial v^i(r)}{\partial r_\omega} < 0.$$

Suppose that all valuation functions are monotonic.[42] For each venture $\omega \in \Omega$ and agent $i \in a(\omega)$, let

$$a^-(\omega) \equiv \left\{ i \in a(\omega) : \frac{\partial v^i(r)}{\partial r_\omega} < 0 \right\},$$
$$a^+(\omega) \equiv \left\{ i \in a(\omega) : \frac{\partial v^i(r)}{\partial r_\omega} > 0 \right\}.$$

We define a set of goods $G \equiv \{n\} \cup \{g^i_\omega : \omega \in \Omega \text{ and } i \in a(\omega)\}$, where n is a numeraire good. We let $c^i \in \mathbb{R}^G$ denote the consumption of agent i.

[42] In particular, combined with our assumption that any efficent allocation $r < r^{\max}$, this implies that for each venture ω there is at least one agent for whom ω is costly.

The initial endowment of goods in the economy is given by

$$e^i_g = \begin{cases} r^{\max}_\omega & g = g^i_\omega \text{ and } i \in a^-(\omega) \\ \mathfrak{K} & g = n \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{K}$ is chosen to be greater than the maximum total value created by all ventures in the economy. The consumption vector c induced by an arrangement $[r; p]$ is given by

$$c^i_g([r;p]) \equiv \begin{cases} r_\omega & g = g^i_\omega \text{ and } i \in a^+(\omega) \\ r^{\max}_\omega - r_\omega & g = g^i_\omega \text{ and } i \in a^-(\omega) \\ \mathfrak{K} - p^i \cdot r & g = n \\ 0 & \text{otherwise.} \end{cases}$$

The preferences of agent i over goods are given by

$$u^i(c^i) \equiv v^i((c^i_{g^i_\omega})_{\omega \in \Omega^+_i}, (r^{\max}_\omega - c^i_{g^i_\omega})_{\omega \in \Omega^-_i}, (0)_{\omega \in \Omega \smallsetminus \Omega_i}) + c^i_n,$$

where $\Omega^+_i = \{\omega \in \Omega : i \in a^+(\omega)\}$ and $\Omega^-_i = \{\omega \in \Omega : i \in a^-(\omega)\}$; note that this utility function is weakly increasing over the consumption space $\Gamma \equiv (\times_{\omega \in \Omega}[0, r^{\max}_\omega]^{G_\omega}) \times [0, |I|\mathfrak{K}]$ where $G_\omega \equiv \{g^i_\psi \in G : \psi = \omega\}$. This utility function over goods induces a demand correspondence

$$E^i((q_g)_{g \in G}) \equiv \underset{\substack{c^i \in \Gamma \\ c^i \cdot q \leq e^i \cdot q}}{\arg\max}\{u^i(c^i)\}.$$

For each venture $\omega \in \Omega$, we define a firm f^ω with production technology

$$Y^\omega \equiv \{((-y)_{g \in \{g^i_\omega : i \in a^-(\omega)\}}, (y)_{g \in \{g^i_\omega : i \in a^+(\omega)\}}, (0)_{g \in G \smallsetminus \{g^i_\omega : i \in a(\omega)\}}) : y \in [0, r^{\max}_\omega]\}.$$

This induces a production correspondence

$$R^\omega((q_g)_{g \in G}) = \underset{y \in Y^\omega}{\arg\max}\{q \cdot y\}$$

for firm f^ω. We assume that the shares of each firm are evenly distributed amongst all agents in the economy.[43]

Unfortunately, the existence of a competitive equilibrium in our setting does not follow from standard results in general equilibrium theory, as agents' utilities are not strongly monotone. Nevertheless, we can prove the existence of general equilibrium in our setting using an argument modified from the standard existence proof (given, for instance, in Chapter 17 of Mas-Colell et al. [30]).

We introduce a Walrasian auctioneer who fixes the price of the numeraire good at 1 and

[43] Since production is linear and, by assumption, efficient allocations are interior, firms' profits will be 0 in equilibrium; hence, the ownership of shares does not affect the outcome.

solves the following maximization problem:

$$A((c^i)_{i\in I},(y^\omega)_{\omega\in\Omega}) \equiv \operatorname*{arg\,max}_{q\in[0,\bar{q}]^G} \left\{ q\cdot \left(\sum_{i\in I} c^i - \left(\sum_{i\in I} e^i + \sum_{\omega\in\Omega} y^\omega \right)\right)\right\},$$

where

$$\bar{q} \equiv 2|G| \left(\max_{\substack{i\in I, g\in G\\ c^i\in\Gamma}} \left\{ \frac{\partial u^i(c^i)}{\partial c^i_g} \right\} \right).$$

Now consider the correspondence

$$\mathcal{K} \equiv (\times_{i\in I} E^i) \times (\times_{\omega\in\Omega} R^\omega) \times A$$

which maps

$$\left((\times_{\omega\in\Omega}[0, r^{\max}_\omega]^{G_\omega}) \times [0, |I|\mathfrak{K}]\right) \times (Y^\omega)_{\omega\in\Omega} \times [0,\bar{q}]^G$$

to itself. It follows from Berge's Theorem that the demand correspondence for each agent, the production correspondence for each firm, and the correspondence induced by the problem of the auctioneer are non-empty, compact-valued, and upper hemicontinuous; hence, $\mathcal{K}$ is non-empty, compact-valued, and upper hemicontinuous. Moreover, each of these correspondences is convex-valued, as each is defined by the maximization of a concave function. Therefore, by Kakutani's fixed point theorem, there must exist a fixed point of $\mathcal{K}$, which we denote by $((\tilde{c}^i)_{i\in I}, (\tilde{y}^\omega)_{\omega\in\Omega}, (\tilde{q}_g)_{g\in G})$

It is immediate that at $((\tilde{c}^i)_{i\in I}, (\tilde{y}^\omega)_{\omega\in\Omega}, (\tilde{q}_g)_{g\in G})$, each agent and firm is optimizing given prices $\tilde{q}$. Moreover, the price $\tilde{q}_g$ of each good must be positive, as for each good there is one agent whose utility in that good is strictly increasing (because of the monotonicity assumption). Furthermore, $\tilde{q}_g \neq \bar{q}$ for all $g\in G$ as otherwise some agent or firm is not optimizing.[44] Hence, since $((\tilde{c}^i)_{i\in I}, (\tilde{y}^\omega)_{\omega\in\Omega}, (\tilde{q}_g)_{g\in G})$ is a fixed point, the fact that $\tilde{q}_g \in (0,\bar{q})$ for all $g\in G$ implies (from the auctioneer's problem) that $\sum_{i\in I} c^i - \left(\sum_{i\in I} e^i + \sum_{\omega\in\Omega} y^\omega\right) = 0$, i.e., markets clear.

A competitive equilibrium of this economy induces a competitive equilibrium of the original multilateral matching economy. We construct an arrangement $[\tilde{r};\tilde{p}]$ by, for each $\omega\in\Omega$, letting

$$\tilde{r}_\omega = r^{\max}_\omega - \tilde{c}^i_{g^i_\omega}$$

[44]Note that the price of every output must be no greater than

$$\max_{\substack{i\in I, g\in G\\ c^i\in\Gamma}} \left\{ \frac{\partial u^i(c^i)}{\partial c^i_g} \right\};$$

hence, the price of every input is bounded above by

$$|G| \left(\max_{\substack{i\in I, g\in G\\ c^i\in\Gamma}} \left\{ \frac{\partial u^i(c^i)}{\partial c^i_g} \right\} \right) < \bar{q}.$$

for some $i \in a^{-}(\omega)$, and

$$\tilde{p}^i_\omega = \tilde{q}_{g^i_\omega}$$

for all $i \in I$. It is clear that $\tilde{r} \in D^i(\tilde{p})$ for each $i \in I$; hence, $[\tilde{r}; \tilde{p}]$ is a competitive equilibrium.

The preceding discussion shows that in the presence of valuation monotonicity, our existence result can be re-derived through an appeal to general equilibrium. However, for multilateral matching models in which valuation functions are not monotonic, it is not generally possible to associate goods with ventures in a fashion which yields the utility monotonicity assumptions that are standard in general equilibrium models. For instance, consider the following non-monotonic valuation function:

$$v^i(r) = r_\omega - \frac{1}{2}(r_\omega)^2. \tag{A.1}$$

If $r_\omega < 1$, then i should be a "buyer" of an "output" associated with ω, but if $r_\omega > 1$, then i should be a "seller" of an "input" associated with ω. Thus, there is no natural general equilibrium economy corresponding to a multilateral matching economy with v^i as in (A.1); in such an economy, the good associated with the participation of i would change from being an input to an output depending on the preferences of other agents in $a(\omega)$.

Appendix B. Proofs Omitted from the Main Text

Proof of Theorem 1

Consider any competitive equilibrium $[r; p]$. Theorem 7 shows that $\kappa([r; p])$ is strongly group stable, hence by Theorem 6 it is in the core and efficient.

Proof of Theorem 2

We consider any efficient $\hat{r}$. By definition, $\hat{r}$ is a solution to the problem

$$\underset{0 \le r \le r^{\max}}{\arg\max} \left\{ \sum_{i \in I} v^i(r) \right\}. \tag{B.1}$$

For each agent $i \in I$, denote by $\partial v^i(r)$ the subgradient of v^i at r. Since the v^i are all continuous, $\partial v^i(r)$ is nonempty for all r.

If $p^i \in \partial v^i(\hat{r})$, then $\hat{r}$ is a solution to

$$\underset{0 \le r \le r^{\max}}{\arg\max} \{ v^i(r) - p^i \cdot r \},$$

as $v^i(r)$ is concave. Thus, to show the result it suffices to show that for each $i \in I$ there exists $p^i \in \partial v^i(\hat{r})$ such that the matrix p is a valid price matrix (i.e., so that $\sum_{i \in I} p^i_\omega = 0$ for all $\omega \in \Omega$). But this is immediate: Since $\hat{r}$ maximizes (B.1), we must have[45]

$$0 \in \partial \sum_{i \in I} v^i(\hat{r}) = \sum_{i \in I} \partial v^i(\hat{r});$$

it follows that there exist $p^i \in \partial v^i(\hat{r})$ such that $\sum_{i \in I} p^i = 0$.

Proof of Theorem 3

Let $\hat{r}$ be a solution to (B.1); such a solution is guaranteed to exist as the v^i are all continuous and the domain of the maximization problem (B.1) is compact. The allocation $\hat{r}$ is efficient and hence, by Theorem 2 (which makes use of our concavity assumption), there exist prices p such that $[\hat{r}; p]$ is a competitive equilibrium.

The uniqueness of the competitive equilibrium in the case when the v^i are strictly concave and continuously differentiable is immediate: Strict concavity implies that there exists a unique $\hat{r}$ solving (B.1). Furthermore, when the valuation functions v^i are continuously differentiable, the subgradients ∂v^i are single-valued, and hence yield a unique price matrix p in the proof of Theorem 2.

Proof of Theorem 4

We suppose that the function $v^i(r)$ is not concave at the point $\tilde{r} \in \times_{\omega \in \Omega}[0, r^{\max}_\omega]$.[46] For each $j \neq i$, we set

$$v^j(r) = -\mathfrak{m}\|r_{\Omega_j} - \tilde{r}_{\Omega_j}\|,$$

where $\|\cdot\|$ is the Euclidean norm and $\mathfrak{m} \in \mathbb{R}_{\geq 0}$ is sufficiently large that $\tilde{r}$ is the unique solution to the global maximization problem (B.1).

By construction, $\tilde{r}$ is the only efficient allocation. However, there do not exist prices p for which $[\tilde{r}; p]$ is a competitive equilibrium. Indeed, for any choice of p^i we have

$$\tilde{r} \notin \operatorname*{arg\,max}_{0 \leq r \leq r^{\max}} \{v^i(r) - p^i \cdot r\},$$

as v^i is not concave at $\tilde{r}$. Hence, $\tilde{r} \notin D^i(p) \supseteq D(p)$ for any p. It then follows from Theorem 1 that no arrangement can be a competitive equilibrium.

Proof of Theorem 5

From Theorem 1, we know that

$$\hat{r}(\ell) = \operatorname*{arg\,max}_{0 \leq r \leq r^{\max}} \left\{ \sum_{i \in I} v^i(r; \ell) \right\}. \tag{B.2}$$

[45] Here, for sets $A \subseteq \mathbb{R}^\Omega$ and $B \subseteq \mathbb{R}^\Omega$, we denote by $A + B$ the *sumset*

$$A + B = \{a + b : a \in A, b \in B\}.$$

[46] Note that this implies that there is least one agent in addition to i who shares participation in some venture in Ω_i.

Taking first-order conditions of the constrained maximization problem (B.2) with respect to r_ω for all $\omega \in \Omega$, we have

$$\sum_{i\in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\omega} + \lambda_\omega - \mu_\omega = 0$$

along with the constraint conditions

$$\begin{aligned} \lambda_\omega(0 - r_\omega) &= 0, \\ \mu_\omega(r_\omega - r^{\max}_\omega) &= 0. \end{aligned}$$

From the implicit function theorem, we have

$$\frac{\partial \hat{r}(\ell)}{\partial \ell} = -\mathbf{H}^{-1} \frac{\partial}{\partial \ell} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i\in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_{\omega^1}} \\ \vdots \\ \sum_{i\in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_{\omega^{|\Omega|-1}}} \\ \sum_{i\in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\psi} \end{pmatrix},$$

where $\mathbf{H}$ is the bordered Hessian of our constrained maximization problem and we have denoted $\Omega = \{\omega^1, \ldots, \omega^{|\Omega|-1}, \psi\}$. Hence,

$$\begin{aligned} \frac{\partial \hat{r}(\ell)}{\partial \ell} &= -\mathbf{H}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i\in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\psi} \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \frac{\partial \hat{r}(\ell)}{\partial \ell} &= -\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \mathbf{H}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i\in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\psi} \end{pmatrix}. \end{aligned}$$

It follows immediately that $\frac{\partial \hat{r}_\psi(\ell)}{\partial \ell} \geq 0$, as $\mathbf{H}$ is negative semidefinite.

Proof of Theorem 7

Since $[r;p]$ is a competitive equilibrium, we have that for all $i \in I$,

$$\begin{aligned} r \in D^i(p) &= \underset{0\leq r \leq r^{\max}}{\arg\max}(v^i(r) - p^i \cdot r), \\ \kappa([r;p])_i &\in \underset{Z\subseteq Y_i}{\arg\max}(v^i(Z) - p^i \cdot \rho(Z)), \\ \kappa([r;p])_i &\in C^i(\kappa([r;p])_i), \end{aligned}$$

where $Y \equiv \{(\omega, \tilde{r}_\omega, \tilde{s}_\omega) \in X : \omega \in \Omega, \tilde{r}_\omega \in [0, r_\omega^{\max}], \tilde{s}_\omega = p_\omega \cdot \tilde{r}_\omega\}$. The last line follows as $\kappa([r;p]) \subseteq Y$. Hence, $\kappa([r;p])$ is individually rational.

Now suppose that $\kappa([r;p])$ is not strongly unblocked, and let Z be a set such that for all $i \in a(Z)$ there exists a $Y^i \subseteq Z \cup \kappa([r;p])$ such that $Z_i \subseteq Y^i$ and $u^i(Y^i) > u^i(\kappa([r;p]))$. For each $i \in a(Z)$, fix a $Y^i \in C^i(Z \cup \kappa([r;p]))$ such that $Z_i \subseteq Y^i$. For all $i \in a(Z)$, we have that

$$\begin{aligned} u^i(Y^i) &> u^i(\kappa([r;p])) \\ v^i(\rho(Y^i)) - \pi^i(Y^i) \cdot \rho(Y^i) &> v^i(\rho(\kappa([r;p]))) - \pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p])). \end{aligned} \tag{B.3}$$

Summing (B.3) over agents $i \in a(Z)$, we obtain

$$\begin{aligned} \sum_{i \in a(Z)} \left(v^i(\rho(Y^i)) - \pi^i(Y^i) \cdot \rho(Y^i)\right) &> \sum_{i \in a(Z)} \left(v^i(\rho(\kappa([r;p]))) - \pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p]))\right), \\ \sum_{i \in a(Z)} \left(v^i(\rho(Y^i)) - p^i \cdot \rho(Y^i)\right) &> \sum_{i \in a(Z)} \left(v^i(\rho(\kappa([r;p]))) - p^i \cdot \rho(\kappa([r;p]))\right), \end{aligned} \tag{B.4}$$

where the second inequality follows as

1. $\pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p])) = p^i \cdot \rho(\kappa([r;p]))$;
2. if $(\omega, \hat{r}_\omega, \hat{s}_\omega) \in Z$, then $a(\omega) \subseteq a(Z)$, hence, as as $(\omega, \hat{r}_\omega, \hat{s}_\omega) \in Y^i$ for all $i \in a(Z)$, we have $\sum_{i \in a(Z)} \pi_\omega^i(Y^i) = 0 = \sum_{i \in a(Z)} p_\omega^i$; and
3. if $(\omega, \hat{r}_\omega, \hat{s}_\omega) \in Y^i \setminus Z$ for some $i \in a(Z)$, then $\hat{s}_\omega = p_\omega^i r_\omega$.

But the inequality (B.4) implies that for at least one $j \in a(Z)$,

$$v^j(\rho(Y^j)) - p^j \cdot \rho(Y^j) > v^j(\rho(\kappa([r;p]))) - p^j \cdot \rho(\kappa([r;p])) = v^j(r) - p^j \cdot r$$

so that $r \notin D^j(p)$ and, hence, $[r;p]$ is not a competitive equilibrium.

Thus, $\kappa([r;p])$ is strongly unblocked and, hence, is strongly group stable. That $\kappa([r;p])$ is stable and in the core then follows from Theorem 6.

Proof of Theorem 8

Consider any stable outcome A. Let $r = \rho(A)$ and $p = \pi(A)$. Let $s_\omega^i = p_\omega^i r_\omega$ be the transfer payment from agent i associated with the contract $(\omega, r_\omega, s_\omega) \in A$. Suppose that r is not efficient. Then as the v^i are concave, we know that $0 \notin \partial \sum_{i \in I} v^i(r)$. It follows that there exists $\psi \in \Omega$ such that $0 \neq \breve{r}_\psi$ for all $\breve{r} \in \partial \sum_{i \in I} v^i(r)$. Choose some $\mathring{r} \in \partial \sum_{i \in I} v^i(r)$, let

$$\check{r}_\omega = \begin{cases} \mathring{r}_\psi & \omega = \psi \\ 0 & \text{otherwise,} \end{cases}$$

and let $\tilde{r} \equiv r + \epsilon \check{r}$, with $\epsilon \neq 0$ chosen so that $\sum_{i \in I} v^i(\tilde{r}) > \sum_{i \in I} v^i(r)$ and $0 \leq \tilde{r} \leq r^{\max}$.[47]

[47]Note that since $0 \notin \partial \sum_{i \in I} v^i(r)$, r is not a global maximum, and in particular since $0 \notin [\partial \sum_{i \in I} v^i(r)]_\psi$, there exists some ϵ such that $\sum_{i \in I} v^i(r) < \sum_{i \in I} v^i(\tilde{r})$; it is clear that $\tilde{r}_\psi \in [0, r_\psi^{\max}]$.

Now consider the set $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\}$ where

$$\tilde{s}^j_\psi \equiv \begin{cases} s^j_\psi + (v^j(\tilde{r}) - v^j(r)) - \frac{\sum_{i \in a(\psi)}[v^i(\tilde{r}) - v^i(r)]}{|a(\psi)|} & j \in a(\psi) \\ 0 & \text{otherwise.} \end{cases}$$

Each agent $j \in a(\psi)$ strictly prefers $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\} \cup (A \setminus \{(\psi, r_\psi, s_\psi)\})$ to A.[48] It follows that $(\psi, \tilde{r}_\psi, \tilde{s}_\psi) \in Y$ for each $Y \in C^j(\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\} \cup A$, and so $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\}$ blocks A.

Proof of Corollary 2

Given Theorem 8, the result follows immediately from Theorem 2, as for any efficient allocation r, we can find prices p such that $[r; p]$ is a competitive equilibrium.

Proof of Lemma 1

We first show that each actor of type $j \in J$ obtains an optimal bundle given t. Suppose not; then there exists a bundle $\bar{G}$ such that $w^j_{\bar{G}} - \sum_{g \in \bar{G}} t_g > w^j_{\tilde{G}} - \sum_{g \in \tilde{G}} t_g$ for some bundle $\tilde{G}$ such that $A^j_{\tilde{G}} > 0$. But then $r_{\omega^j_{\tilde{G}}} > 0$, and hence

$$\tilde{r}_\omega = \begin{cases} r_{\omega^j_{\bar{G}}} + r_{\omega^j_{\tilde{G}}} & \omega = \omega^j_{\bar{G}} \\ 0 & \omega = \omega^j_{\tilde{G}} \\ r_\omega & \text{otherwise} \end{cases}$$

provides a higher utility than r for agent j under prices $\bar{p}$, contradicting the assumption that $[r; \bar{p}]$ is a competitive equilibrium. Thus, we see that, under A, each actor obtains an optimal bundle t.

Finally, we show that A is a feasible allocation. First, we note that since $r \geq 0$, $A^j_{\bar{G}} \geq 0$ for all $\bar{G} \neq \varnothing$ for each $j \in J$. Second, we observe that $A^j_\varnothing \geq 0$, as $\sum_{\omega \in \Omega_j} r_\omega \leq m^j$ follows from (3).[49] Lastly, we conclude that, for each good $g \in G$,

$$\begin{aligned} \sum_{j \in J} m^j \left(\sum_{\{g\} \subseteq \bar{G} \subseteq G} A^j_{\bar{G}} \right) &= \sum_{j \in J} m^j \left(\sum_{\{g\} \subseteq \bar{G} \subseteq G} \frac{r_{\omega^j_{\bar{G}}}}{m^j} \right) \\ &= \sum_{j \in J} \sum_{\{g\} \subseteq \bar{G} \subseteq G} r_{\omega^j_{\bar{G}}} \\ &= \sum_{\omega \in \Omega_g} r_\omega \\ &= q^g, \end{aligned}$$

where the last equality follows from (2).

[48] It may be the case that there does note exist a contract $(\psi, r_\psi, s_\psi) \in A$; in that case, we have that $j \in a(\psi)$ strictly prefers $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\} \cup A)$ to A.

[49] Note that this also implies that $A^j_{\bar{G}} \leq 1$ for all $j \in J$ and $\bar{G} \subseteq G$.

Proof of Theorem 10

We let

$$F(\tilde{r}) \equiv \underset{0 \le r \le r^{\max}}{\arg\max} \left\{ \sum_{i \in I} v^i(r_{\Omega_i}; \tilde{r}_{\Omega - \Omega_i}) \right\}.$$

Note that by the Theorem of the Maximum, F is non-empty, compact-valued, and upper hemicontinuous. Moreover, each of these correspondences is convex-valued, as each is defined by the maximization of a concave function. As $\times_{\omega \in \Omega}[0, r^{\max}_\omega]$ is non-empty, compact, and convex, Kakutani's fixed point theorem implies that there exists an $\hat{r}$ such that $F(\hat{r}) = \hat{r}$.

An argument exactly analogous to the proof of Theorem 2 then shows that there exists a price matrix p such that

$$\hat{r} \in \bar{D}^i(p; \hat{r});$$

hence, $[\hat{r}; p]$ is a competitive equilibrium.

Appendix C. A Fixed-Point Approach to Theorem 3

In this appendix, we provide an alternative proof of our competitive equilibrium existence result using a technique similar to that used in proving the existence of competitive equilibria in general equilibrium theory. For simplicity, we assume in this section that, for each $i \in I$, the valuation function v^i is differentiable.

We let

$$\bar{p} \equiv 2 \max \left\{ \left| \frac{\partial v^i(r)}{\partial r_\omega} \right| : i \in I,\ \omega \in \Omega,\ \text{and } 0 \le r \le r^{\max} \right\},$$

be a price so high that no agent will demand any venture at that price. We now introduce an "auctioneer" who maximizes the cost of under-demanded and over-demanded venture participation. First, we define the *price space* as

$$\Delta \equiv \left\{ p \in [-\bar{p}, \bar{p}]^{I \times \Omega} : \text{For all } \omega \in \Omega, \sum_{j \in a(\omega)} p^j_\omega = 0 \text{ and } p^i_\omega = 0 \text{ for all } i \in I \smallsetminus a(\omega) \right\}.$$

The optimization problem A of the auctioneer is

$$A^i_\omega(\{r^k\}_{k \in I}) \equiv \underset{p \in \Delta}{\arg\max} \left\{ p^i_\omega \left(r^i_\omega - \frac{1}{|I|} \sum_{j \in a(\omega)} r^j_\omega \right) \right\}.$$

It follows from Berge's Theorem that A is a non-empty, compact-valued, and upper hemicontinuous correspondence. Furthermore, it also follows from Berge's Theorem that the demand correspondence for each agent is non-empty, compact-valued, and upper hemicontinuous. Moreover, each of these correspondences is convex-valued, as each is defined by the maximization of a concave function. Hence, the correspondence

$$(\times_{i \in I} D^i) \times A : (\times_{\omega \in \Omega}[0, r^{\max}_\omega])^I \times [-\bar{p}, \bar{p}]^{I \times \Omega} \rightrightarrows (\times_{\omega \in \Omega}[0, r^{\max}_\omega])^I \times [-\bar{p}, \bar{p}]^{I \times \Omega} \quad \text{(C.1)}$$

has a fixed point $(\{r^i\}_{i\in I}, p)$ by Kakutani's fixed point theorem. Furthermore, since $(D^k(p))_\omega = [0, r_\omega^{\max}]$ for all $k \in I \smallsetminus a(\omega)$, the point $(\{\tilde{r}^i\}_{i\in I}, p)$ where $\tilde{r}^j_\omega = r^j_\omega$ for all $j \in a(\omega)$ and $\tilde{r}^j_\omega = \frac{1}{|I|}\sum_{i\in a(\omega)} r^i_\omega$ for all $j \in I \smallsetminus a(\omega)$ is also a fixed point of (C.1).

It follows from the definition of A that $r^i_\omega = r^j_\omega$ for all $i, j \in a(\omega)$, since $|p^i_\omega| < \bar{p}$ for all $i \in I$ and $\omega \in \Omega$.[50] Hence $\tilde{r}^i_\omega = \tilde{r}^j_\omega \equiv \tilde{r}_\omega$ for all $i, j \in I$ and $\omega \in \Omega$. Hence, the arrangement $[\tilde{r}; p]$ is a competitive equilibrium, as each agent is demanding an optimal participation vector given prices and each agent demands the same level of participation for each venture.

Appendix D. Example of a Core and Stable Outcome That Is Not Strongly Group Stable

We present an example of an outcome that is in the core and stable, but is not strongly group stable. Let $I = \{i, j\}$, $\Omega = \{\chi, \psi, \omega\}$, $a(\chi) = a(\psi) = a(\omega) = I$, $r_\chi^{\max} = r_\psi^{\max} = r_\omega^{\max} = 1$, and

$$\begin{aligned} v^i(r) &= -2r_\psi - 2r_\omega - 5\min\{r_\psi, r_\omega\} - 11\min\{r_\chi, r_\psi, r_\omega\}, \\ v^j(r) &= 2r_\chi + r_\psi + r_\omega + 11\min\{r_\chi, r_\psi, r_\omega\}. \end{aligned}$$

Any outcome of the form $A = \{(\chi, 1, (-q, q))\}$ such that $0 \le q \le 2$ is both stable and in the core. However, no such A is strongly group stable, as no such A is strongly unblocked—to see this, take $Z = \{(\psi, 1, (-6, 6)), (\omega, 1, (-6, 6))\}$.

Appendix E. The Illustrative Example Revisited

In this appendix, we provide the underlying production economy for the example of Section 2, under the additional assumption that $(r_\omega^{\max}, r_\psi^{\max}) = (400, 210)$. Let $I = \{c, s, g, k, b\}$, $\mathsf{G} = \{\mathsf{c}, \mathsf{g}, \mathsf{s}, \mathsf{k}\}$, and $\Omega = \{\omega, \psi\}$ where the production processes are defined by

$$\omega^i_{\mathsf{h}} = \begin{cases} -\frac{1}{5} & i = c \text{ and } \mathsf{h} = \mathsf{c} \\ -\frac{2}{5} & i = g \text{ and } \mathsf{h} = \mathsf{g} \\ -\frac{2}{5} & i = s \text{ and } \mathsf{h} = \mathsf{s} \\ 1 & i = k \text{ and } \mathsf{h} = \mathsf{k} \\ 0 & \text{otherwise,} \end{cases} \qquad \psi^i_{\mathsf{h}} = \begin{cases} -1 & i = g \text{ and } \mathsf{h} = \mathsf{g} \\ 1 & i = b \text{ and } \mathsf{h} = \mathsf{g} \\ 0 & \text{otherwise,} \end{cases}$$

[50] If for some $i \in I$ and $\omega \in \Omega$, $p^i_\omega = \bar{p}$, then, by assumption on the size of $\bar{p}$, the demand of i for venture ω by i must be 0. But then p must not be an optimal solution to the problem of the auctioneer, implying that $(\{r^i_\omega\}_{i\in I}, p)$ is not a fixed point. Analogous reasoning holds if $p^i_\omega = -\bar{p}$.

consumption valuations are given by

$$\dot{v}^c(c^c) = 80c^c_{\mathsf{c}},$$
$$\dot{v}^g(c^g) = 25c^g_{\mathsf{g}},$$
$$\dot{v}^s(c^s) = 25c^s_{\mathsf{s}} - \frac{1}{16}(c^s_{\mathsf{s}})^2,$$
$$\dot{v}^k(c^k) = 60c^k_{\mathsf{k}} - \frac{7}{100}(c^k_{\mathsf{k}})^2,$$
$$\dot{v}^b(c^b) = 32\max\{c^b_{\mathsf{g}}, 50\},$$

and initial endowments are

$$e^c = (80, 0, 0, 0),$$
$$e^g = (0, 210, 0, 0),$$
$$e^s = (0, 0, 160, 0),$$
$$e^k = (0, 0, 0, 0),$$
$$e^b = (0, 0, 0, 0).$$

(We use the convention that the elements of vector $e^i = (e^i_{\mathsf{c}}, e^i_{\mathsf{g}}, e^i_{\mathsf{s}}, e^i_{\mathsf{k}})$ are given in the order cement, gravel, sand, concrete.) It is clear that the production economy thus illustrated yields the multilateral matching economy presented in Section 2.

Appendix F. Example of an Inefficient Competitive Equilibrium in the Presence of Externalities

We present an example of an inefficient competitive equilibrium in the presence of externalities. Let $I = \{i, j, k\}$, $\Omega = \{\psi, \omega\}$, $a(\psi) = \{i, j\}$, $a(\omega) = \{i, k\}$, $r^{\max}_{\psi} = r^{\max}_{\omega} = 1$, and

$$v^i(r) = -r_\psi^2 - r_\omega^2,$$
$$v^j(r) = (1 - r_\psi - r_\omega)r_\psi,$$
$$v^k(r) = (1 - r_\psi - r_\omega)r_\omega.$$

The unique efficient allocation is $(\frac{1}{6}, \frac{1}{6})$, with total surplus $\frac{1}{6}$. However, the unique competitive equilibrium is

$$\left[\left(\frac{1}{5}, \frac{1}{5}\right); \begin{pmatrix} -\frac{2}{5} & \frac{2}{5} & 0 \\ -\frac{2}{5} & 0 & \frac{2}{5} \end{pmatrix}\right],$$

with total surplus $\frac{4}{25} < \frac{1}{6}$.

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