# Vacancies in Supply Chain Networks*

John William Hatfield
Graduate School of Business
Stanford University

Scott Duke Kominers
Harvard Business School
Harvard University

June 22, 2011

**Abstract**

We use the supply chain matching framework to study the effects of firm exit. We show that the exit of an initial supplier or end consumer has monotonic effects on the welfare of initial suppliers and end consumers, but may simultaneously have positive and negative effects on intermediaries. Furthermore, we demonstrate that there are no clear comparative statics for the effects of removing an intermediary on the welfare of other firms; most surprisingly, removing an intermediary may diminish the welfare of other firms at the same level of the supply chain.

*JEL classification:* C78; L14

*Keywords*: Matching; Networks; Stability; Vacancy Chains

*We are grateful to Daron Acemoglu, Drew Fudenberg, Sonia Jaffe, Fuhito Kojima, Assaf Romm, Alvin E. Roth, and workshop participants at Harvard for helpful comments. Hatfield appreciates the hospitality of Harvard Business School, which hosted him during parts of this research. Kominers gratefully acknowledges the support of a National Science Foundation Graduate Research Fellowship, a Yahoo! Key Scientific Challenges Program Fellowship, and a Terence M. Considine Fellowship in Law and Economics funded by the John M. Olin Center at Harvard Law School. Any suggestions are welcome and may be emailed to hatfield@stanford.edu or skominers@hbs.edu.

# 1 Introduction

In 2008, Ford Motor Company President and CEO Alan R. Mulally (2008) testified before Congress, advocating for a bailout of Ford's direct competitors General Motors and Chrysler. This behavior at first seems difficult to reconcile with economic theory—why should Ford plead for the survival of its direct competitors?[1,2] However, as we show in this note, this behavior can arise naturally when intermediate producers in supply chains networks have preferences over suppliers.[3]

We model the effect of exit from supply chain networks using the supply chain matching model of Ostrovsky (2008). We demonstrate two contrasting results: The exit of an end consumer benefits other end consumers while harming the initial suppliers at the head of the supply chain.[4] Meanwhile, there are no clear comparative statics for the welfare effects of removing an intermediate producer on other intermediaries, initial suppliers, and end consumers.[5] In particular, contrary to standard intution regarding the loss of competitors, removing an intermediary may diminish the welfare of other firms at the same level of the supply chain.

Our results sharpen Theorem 3 of Ostrovsky (2008), which shows that without a given initial supplier in the market, the best and worst stable outcomes for other initial suppliers improve, while those for end consumers worsen. The Ostrovsky (2008) result only compares the extremal stable outcomes in a market with and without a given supplier; in contrast, by studying the process of market reequilibration following firm exit, we may characterize the effect of initial supplier exit on any given stable outcome.

[1] At the time there was significant concern that, without government action, General Motors and Chrysler could be forced to liquidate (Isidore (2008)). Thus, it seems likely that, without government action, General Motors and Chrysler would (at least) have become weaker competitors for Ford.

[2] We are indebted to Daron Acemoglu for this example. Acemoglu et al. (2010) give an alternative explanation of Ford's behavior, focusing on issues of aggregate volatility in supply chain networks.

[3] Such preferences arise whenever firm interactions involve relationship-specific capital (Williamson (1983)). Relationship-specific capital has been identified, e.g., in manufacturing (Parsons (1972)) and coal markets (Joskow (1987)).

[4] By symmetry, an analogous result holds for the effects of an initial supplier's exit.

[5] Similar analysis shows that there are no clear comparative statics for the effects of initial supplier (or end consumer) exit on intermediary welfare.

Our work follows in the tradition of "vacancy chain" results for matching markets. We show that the vacancy chain results of Gale and Sotomayor (1985), Blum et al. (1997), and Hatfield and Milgrom (2005) generalize to supply chain networks, but only in a very specific sense—they apply only to firms at the ends of the supply chain, and not to intermediaries.[6] These observations underscore the importance of relation-specific contracting in supply chain dynamics.

# 2 Model

We begin by introducing the standard supply chain matching framework of Ostrovsky (2008), using the notation of Hatfield and Kominers (2010b); readers familiar with matching theory may wish to skip to Section 3.

There is finite set $F$ of of firms, and a finite set $X$ of contracts. Each contract $x \in X$ is associated with both a buyer $x_B$ and a seller $x_S$; there may be several contracts with the same buyer and the same seller. For notational convenience, we let $Y|_f \equiv \{y \in Y : f \in \{y_B, y_S\}\}$ denote the set of contracts in $Y$ associated with firm $f$; we extend this notation by writing $Y|_G = \cup_{g \in G}(Y|_g)$ for $G \subseteq F$.

We assume that the contract set $X$ is **acyclic**, i.e. that there does not exist a set of contracts

$$\{x^1, \ldots, x^N\} \subseteq X$$

such that $x^1_B = x^2_S, x^2_B = x^3_S, \ldots, x^{N-1}_B = x^N_S, x^N_B = x^1_S$. This assumption is equivalent to the assumption of **supply chain structure**, i.e. the existence of an ordering $\rhd$ on $F$ such that for all $x \in X$, $x_S \lhd x_B$.

[6] Our positive result also applies in more restricted settings for which vacancy chain results have not previously been proven, such as the settings of many-to-many matching (Echenique and Oviedo (2006)) and many-to-many matching with contracts (Klaus and Walzl (2009); Hatfield and Kominers (2010a)).

## Preferences

Each $f \in F$ has a strict preference relation $P^f$ over sets of contracts involving $f$. For any $Y \subseteq X$, we first define the **choice set** of $f$ as the set of contracts $f$ chooses from $Y$. We define

$$C^f(Y) \equiv \max_{P^f}\{Z \subseteq Y : x \in Z \Rightarrow f \in \{x_B, x_S\}\}.^7$$

The purchase contracts chosen by $f$ from $Y \subseteq X$, given access to sale contracts in $Z \subseteq X$ are recorded by:

$$C^f_B(Y|Z) \equiv \{x \in C^f(\{y \in Y : y_B = f\} \cup \{z \in Z : z_S = f\}) : x_B = f\}.$$

Analogously, we define

$$C^f_S(Z|Y) \equiv \{x \in C^f(\{y \in Y : y_B = f\} \cup \{z \in Z : z_S = f\}) : x_S = f\}.$$

We also define the **rejected set** of contracts when acting as a buyer or as a seller as

$$R^f_B(Y|Z) \equiv Y - C^f_B(Y|Z),$$
$$R^f_S(Z|Y) \equiv Z - C^f_S(Z|Y).$$

Let $C_B(Y|Z) \equiv \bigcup_{f \in F} C^f_B(Y|Z)$ be the set of contracts chosen from $Y$ by some firm as a buyer, and $C_S(Z|Y) \equiv \bigcup_{f \in F} C^f_S(Z|Y)$ be the set of contracts chosen from $Z$ by some firm as a seller. Let $R_B(Y|Z) \equiv Y - C_B(Z|Y)$ and $R_S(Z|Y) \equiv Z - C_S(Z|Y)$.

The preferences of $f \in F$ are **same-side substitutable** if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

1. $R^f_B(Y'|Z) \subseteq R^f_B(Y|Z)$ and

[7] Here, we use the notation $\max_{P^f}$ to indicate that the maximization is taken with respect to the preferences of firm $f$.

2. $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$.

Similarly, the preferences of $f \in F$ are **cross-side complementary** if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

1. $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$ and
2. $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$.

If a firm's preferences are both same-side substitutable and cross-side complementary, then the firm has **fully substitutable** preferences: The firm is more willing to enter into a contract as a buyer if either there are fewer purchase opportunities available (same-side substitutability), or there are more sale opportunities available (cross-side complementararity). Similarly, the firm is more willing to enter into a contract as a seller if either there are fewer other sale opportunities available (same-side substitutability), or there are more purchase opportunities available (cross-side complementarity).

## Stability

An **outcome** is a set of contracts $A \subseteq X$. An outcome is **stable** if it is

1. **Individually rational**: for all $f \in F$, $C^f(A) = A|_f$;
2. **Unblocked**: There does not exist a nonempty **blocking set** $Z \subseteq X$ such that $Z \not\subseteq A$ and for all $f \in Z_F$, $Z|_f \subseteq C^f(A \cup Z)$.

Stability is the standard solution concept of matching theory (Roth and Sotomayor (1990); Hatfield and Milgrom (2005)). In the presence of fully substitutable preferences, it is equivalent to the *chain stability* solution concept studied by Ostrovsky (2008); moreover, it is known in that case that stable outcomes always exist (Ostrovsky (2008); Hatfield and Kominers (2010b)).

# 3 Vacancy Dynamics

To formally study the effects of market exit in the supply chain matching model established above, we first introduce the following generalized deferred acceptance operator $\Phi^G$, which tracks contract offers made after the firms in $G \subseteq F$ leave the market:

$$\Phi_B^G(X^B, X^S) \equiv X - (R_S(X^S|X^B) \cup (X|_G))$$
$$\Phi_S^G(X^B, X^S) \equiv X - (R_B(X^B|X^S) \cup (X|_G))$$
$$\Phi^G(X^B, X^S) \equiv (\Phi_B^G(X^B, X^S), \Phi_S^G(X^B, X^S)).$$

For any input $(X^B, X^S)$ to the operater $\Phi^G$, we say that $X^B$ and $X^S$ are **buyer** and **seller offer sets** associated with the outcome $X^B \cap X^S$. Note that at each iteration of $\Phi^G$ all offers made to firms in $G$ (i.e. contracts in $(X^B \cup X^S) \cap (X|_G)$) are removed.

When firms' preferences are fully substitutable, iteration of the operator $\Phi^G$ on input $(X^B, X^S)$ leads to a fixed point $\tilde{\Phi}^G(X^B, X^S)$; moreover, for any fixed point $(X^B, X^S)$ of $\Phi^G$, the outcome $X^B \cap X^S$ associated with $(X^B, X^S)$ is a stable outcome of the economy with firms $F - G$ and contract set $X|_{F-G}$ (Hatfield and Kominers (2010b)).

We model the **exit** of firms $G \subseteq F$ from the economy as a transition from the economy with firm set $F$ and contract set $X$ to the economy with firm set $F - G$ and contract set $X|_{F-G}$. Following the exit of $G \subseteq F$, the dynamics of the market readjustment from a stable outcome $A$ associated with offer sets $X^B$ and $X^S$ follow the running of the deferred acceptance operator $\Phi^G$ starting with input $(X^B|_{F-G}, X^S|_{F-G})$; that is, following the exit of $G$ from the economy stabilized at $A$, the market restabilizes at the stable outcome associated with $\tilde{\Phi}^G(X^B, X^S)$.

Under these vacancy dynamics, the impact of a firm's exit depends on that firm's position in the supply chain. To see this, we seperately consider firms which are

1. **Initial Suppliers**: $f \in F$ such that for all $x \in X|_f$, $f = x_S$;

2. **End Consumers**: $f \in F$ such that for all $x \in X|_f$, $f = x_B$;

3. **Intermediaries**: $f \in F$ which are neither initial suppliers nor end consumers.

We obtain the following theorem characterizing the effect of an end consumer's exit; an analogous result holds for the exit of an initial supplier.[8]

**Theorem.** *Suppose that all firms' preferences are fully substitutable and that $A$ is a stable outcome with associated buyer and seller offer sets $X^B$ and $X^S$. Suppose that an end consumer $b$ leaves the market, and let $(\hat{X}^B, \hat{X}^S) \equiv \tilde{\Phi}^{\{b\}}(X^B, X^S)$, with associated outcome $\hat{A} \equiv \hat{X}^B \cap \hat{X}^S$, be the result of the market readjustment process. Then, all initial producers weakly prefer $A$ to $\hat{A}$ and all end consumers (other than $b$) weakly prefer $\hat{A}$ to $A$.*

To see the intution behind this result, consider a firm $f$ that loses an opportunity to sell to the end consumer $b$. Given the loss of $b$, $f$ may wish to accept a previously-rejected offer to sell to a firm $g$; $f$ may also wish to reject a previously-accepted offer to buy from a firm $h$. This, in turn, may lead $g$ and $h$ to accept previously-rejected sale offers and reject previously-accepted purchase offers. Iterating this argument, we see that at each step of the market readjustment process, each firm has (weakly) more purchase offers and weakly fewer sale offers; the theorem follows.

Our theorem generalizes the analagous vacancy chain results of Gale and Sotomayor (1985), Blum et al. (1997), and Hatfield and Milgrom (2005).[9] It also implies Theorem 3 of Ostrovsky (2008), and applies in more restricted settings for which vacancy chain results have not previously been proven (e.g., many-to-many matching (Echenique and Oviedo (2006)) and many-to-many matching with contracts (Klaus and Walzl (2009); Hatfield and Kominers (2010a))). However, as we now show, the earlier vacancy chain results do not generalize beyond our theorem.

[8]The proof of the vacancy chain result for initial supplier exit is analogous to the proof of the result for end consumers.

[9]Kelso and Crawford (1982) study similar dynamics. Mo (1988), Roth and Sotomayor (1990), and Romm (2011) show in increasingly general models that when an agent exits, the welfare of certain other agents must improve, irrespective of the restabilization dynamics.

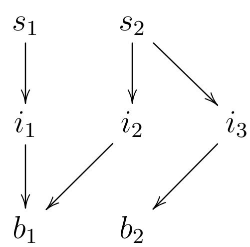
## Effects of Removing an Intermediary

We first demonstrate that eliminating an intermediary may make other intermediates worse off after market readjustment. Consider the following example economy.

**Example Economy.** Let the set of firms be given by $F = \{s_1, s_2, i_1, i_2, i_3, b_1, b_2\}$, where $s_1$ and $s_2$ are initial suppliers, $i_1$, $i_2$, and $i_3$ are intermediaries, and $b_1$ and $b_2$ are end consumers. As depicted below, the set of contracts takes the form

$$X = \{(s_1, i_1), (s_2, i_2), (s_2, i_3), (i_1, b_1), (i_2, b_1), (i_3, b_2)\}$$

where each ordered pair $(f, g) \in X$ represents a contract for which $f$ is the buyer and $g$ is the seller.




Firms' preferences are given by:

$$P^{s_1} : \{(s_1, i_1)\},$$

$$P^{s_2} : \{(s_2, i_3)\} \succ \{(s_2, i_2)\},$$

$$P^{i_1} : \{(s_1, i_1), (i_1, b_1)\},$$

$$P^{i_2} : \{(s_2, i_2), (i_2, b_1)\},$$

$$P^{i_3} : \{(s_2, i_3), (i_3, b_2)\},$$

$$P^{b_1} : \{(i_2, b_1)\} \succ \{(i_1, b_1)\},$$

$$P^{b_2} : \{(i_3, b_2)\}.$$

In the example economy, the only stable outcome is $A = \{(s_1, i_1), (s_2, i_3), (i_1, b_1), (i_3, b_2)\}$.

However, once $i_3$ leaves the market, the only stable outcome is $\hat{A} = \{(s_2, i_2), (i_2, b_1)\}$. Intermediary $i_1$ is worse off after $i_3$ leaves; that is, $i_1$ prefers $A$ to $\hat{A}$. Meanwhile, $i_2$ is clearly better off. Additionally, the outcome for $b_1$ improves when $i_3$ leaves the market, while the outcome for $b_2$ worsens.

This example illustrates that that there is no clear comparative static for intermediary or buyer welfare following the departure of an intermediary.[10] An analogous example can be used to show that there is also no clear comparative static result for seller welfare. These conclusions would hold even if the class of intermediaries were narrowed to include firms which only buy from initial suppliers and only sell to end consumers.

Note that this example can rationalize behavior of the type observed prior to the bailout of General Motors: If $i_3$ (General Motors) is forced out of the market, then its supplier $s_2$ instead supplies $i_2$ (Toyota). This allows $i_2$ to compete more fiercely with $i_1$ (Ford), rendering $i_1$ worse off. Consumer $b_1$ benefits from this increased competition, while consumer $b_2$ is worse off as her favorite intermediary has left the market.

## Effects of End Consumer Exit on Intermediaries

The example economy described in the previous section also illustrates that there is no clear comparative static for intermediary welfare following the departure of a end consumer. Indeed, suppose that $b_2$ exits the market. In that case, the only stable outcome is again $\hat{A} = \{(s_2, i_2), (i_2, b_1)\}$. As expected, intermediary $i_3$ (who sells to $b_2$) is worse off after $b_2$ exits. However, intermediary $i_2$ is better off following the exit of $b_2$, as then $i_3$ no longer wishes to procure the services of $s_2$, and so $s_2$ becomes willing to supply $i_2$.

[10]Ostrovsky (2008) makes a related observation regarding the difference between extremal outcomes in a market with a given intermediary and those in the market without that intermediary.

# 4 Discussion

We have generalized previous vacancy chain results to the context of supply chain matching, showing that the exit of an end consumer (weakly) improves the welfare of all other end cosunmers while simultaneously (weakly) reducing the welfare of initial suppliers. However, as we have demonstrated, there are no clear comparative statics for the effects of an intermediary's exit on the welfare of other firms. The presence of preferences over contracting partners, rather than over just the goods traded is essential for this negative conclusion.

While relationship-specific preferences are not present in all markets, they are a natural consequence of relationship-specific capital (Williamson (1983)). Our work shows that such preferences can vitiate standard intuitions regarding the effects of entry and exit of intermediate producers; economists should therefore be conscious of these issues, and explicitly model relationship-specific preferences when studying the effects of entry and exit.

## References

**Acemoglu, Daron, Asuman Ozdaglar, and Alireza Tahbaz-Salehi**, "Cascades in Networks and Aggregate Volatility," Technical Report, National Bureau of Economic Research 2010.

**Blum, Yosef, Alvin E. Roth, and Uriel G. Rothblum**, "Vacancy chains and equilibration in senior-level labor markets," *Journal of Economic Theory*, 1997, *76* (2), 362–411.

**Echenique, Federico and Jorge Oviedo**, "A theory of stability in many-to-many matching markets," *Theoretical Economics*, 2006, *1*, 233–273.

**Gale, David and Marilda A. O. Sotomayor**, "Some remarks on the stable matching problem," *Discrete Applied Mathematics*, 1985, *11* (3), 223–232.

**Hatfield, John William and Paul Milgrom**, "Matching with Contracts," *American Economic Review*, 2005, *95*, 913–935.

_ **and Scott Duke Kominers**, "Contract Design and Stability in Matching Markets," 2010. Mimeo, Harvard Business School.

_ **and** _ , "Matching in Networks with Bilateral Contracts," 2010. Stanford Graduate School of Business Working Paper 2050.

**Isidore, Chris**, "Consumers to Suffer if GM Goes Under," Special Report on Detroit's Downfall, *Money.CNN.com*, 2008.

**Joskow, Paul L.**, "Contract duration and relationship-specific investments: Empirical evidence from coal markets," *American Economic Review*, 1987, *77* (1), 168–185.

**Kelso, Alexander S. and Vincent P. Crawford**, "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 1982, *50*, 1483–1504.

**Klaus, Bettina and Markus Walzl**, "Stable Many-to-Many Matchings with Contracts," *Journal of Mathematical Economics*, 2009, *45* (7-8), 422–434.

**Mo, Jie-Ping**, "Entry and structures of interest groups in assignment games," *Journal of Economic Theory*, 1988, *46* (1), 66–96.

**Mulally, Alan R.**, "Examining the state of the domestic automobile industry," Hearing, United States Senate Committee on Banking, Housing, and Urban Affairs, 2008.

**Ostrovsky, Michael**, "Stability in supply chain networks," *American Economic Review*, 2008, *98*, 897–923.

**Parsons, Donald O.**, "Specific human capital: An application to quit rates and layoff rates," *Journal of Political Economy*, 1972, *80* (6), 1120–1143.

**Romm, Assaf**, "What Happens when Agents Join an Existing Matching Market? – Extensions for Many-to-One Matching Markets," 2011. Mimeo.

**Roth, Alvin E. and Marilda A. O. Sotomayor**, *Two-sided matching: a study in game-theoretic modeling and analysis*, Cambridge University Press, 1990.

**Williamson, Oliver E.**, "Credible commitments: Using hostages to support exchange," *American Economic Review*, 1983, *73* (4), 519–540.

# Appendix

## Proof of Theorem

We observe that $\Phi_S^{\{b\}}(X^B, X^S) \subseteq \Phi_S^{\varnothing}(X^B, X^S)$ and that $\Phi_B^{\{b\}}(X^B, X^S) = \Phi_B^{\varnothing}(X^B, X^S)$. As firms' preferences are fully substitutable, the rejection functions $R_S$ and $R_B$ are isotone with respect to set inclusion, and hence $\Phi^{\{b\}}$ is isotone with respect to the order $\sqsubseteq$ on $X \times X$ defined by

$$(\dot{X}^B, \dot{X}^S) \sqsubseteq (\bar{X}^B, \bar{X}^S) \iff \dot{X}^B \subseteq \bar{X}^B \text{ and } \dot{X}^S \supseteq \bar{X}^S.$$

Hence, $\Phi^{\{b\}}(X^B, X^S) \sqsupseteq (X^B|_{F-\{b\}}, X^S)$ and so $(\hat{X}^B, \hat{X}^S) = \tilde{\Phi}^{\{b\}}(X^B, X^S) \sqsupseteq (X^B|_{F-\{b\}}, X^S)$. The result then follows as each end consumer $b' \neq b$ prefers $C^{b'}(\hat{X}^B)$ to $C^{b'}(X^B)$ as $\hat{X}^B|_{b'} \supseteq X^B|_{b'}$ and each initial supplier $s'$ prefers $C^{s'}(X^S)$ to $C^{s'}(\hat{X}^S)$ as $\hat{X}^S|_{s'} \subseteq X^S|_{s'}$.