

# Stability, Strategy-Proofness, and Cumulative Offer Mechanisms\*

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## Abstract

We characterize when a stable and strategy-proof mechanism is guaranteed to exist in the setting of many-to-one matching with contracts. We introduce three novel conditions—observable substitutability, observable size monotonicity, and non-manipulability via contractual terms—and show that when these conditions are satisfied, the cumulative offer mechanism is the unique mechanism that is stable and strategy-proof (for workers). Moreover, we show that our three conditions are, in a sense, necessary: If the choice function of some firm fails any of our three conditions, we can construct unit-demand choice functions for the other firms such that no stable and strategy-proof mechanism exists. Thus, our results provide a rationale for the ubiquity of cumulative offer mechanisms in practice.

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*Keywords:* Matching with contracts, Stability, Strategy-proofness, Substitutability, Size monotonicity, Cumulative offer mechanism

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# 1 Introduction

Recently, market design theorists have proposed stable and strategy-proof mechanisms for a wide range of many-to-one matching with contracts settings.<sup>1</sup> [Hatfield and Milgrom \(2005\)](#) showed that when firms’ preferences satisfy the *substitutability* and *size monotonicity* conditions, the worker-proposing cumulative offer mechanism is stable and strategy-proof ([Kelso and Crawford, 1982](#); [Fleiner, 2003](#)).<sup>2</sup> However, in many real-world applications of matching with contracts, the substitutability condition fails, and yet stable and strategy-proof matching is still possible—for example, in

1. entry-level labor markets with regional caps, such as medical-residency matching in Japan (see [Kamada and Kojima \(2012, 2015, 2017, 2018\)](#); see also [Hatfield et al. \(2017\)](#));
2. matching of cadets at West Point and in the Reserve Officer Training Corps to branches of service ([Sönmez and Switzer, 2013](#); [Sönmez, 2013](#); [Jagadeesan, 2019](#));
3. the allocation of airline seat upgrades ([Kominers and Sönmez, 2016](#));
4. the assignment of legal and teaching traineeships in Germany (see [Dimakopoulos and Heller \(2019\)](#) and [Hatfield and Kominers \(2019\)](#), respectively);
5. the placement of students into graduate degrees in psychology in Israel ([Hassidim et al., 2017](#));
6. the matriculation of students into the Indian Institutes of Technology ([Aygün and Turhan, 2017, 2019](#)); and
7. interdistrict school choice programs ([Hafalir et al., 2019](#)).

In the original matching with contracts model, and in all of the settings just described, cumulative offer mechanisms—ascending auction-like mechanisms in which agents on one

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<sup>1</sup>A mechanism is stable if it always produces an outcome in which agents cannot gain from recontracting. A mechanism is strategy-proof for an agent if truth-telling is a dominant strategy for that agent; here, whenever we say that a mechanism is strategy-proof, we mean that the mechanism is strategy-proof for agents on the side of the market with unit demand. It is well-known that no stable mechanism can be strategy-proof for agents who can engage in more than one partnership ([Roth, 1982](#)).

<sup>2</sup>Substitutability requires that whenever the set of contracts available to a hospital expands (in the superset sense), the set of contracts rejected by that hospital also expands. Size monotonicity requires that whenever the set of contracts available to a hospital expands (in the superset sense), the number of contracts chosen by the hospital weakly increases. [Hatfield and Milgrom \(2005\)](#) refer to size monotonicity as the “Law of Aggregate Demand.”

side of the market successively propose contracts to the other side—have been found to be stable and strategy-proof.

In the setting of many-to-one matching with contracts, each of a finite number of doctors desires to sign at most one contract with one of a finite number of hospitals.<sup>3</sup> In a *cumulative offer mechanism*, the outcome is computed via an algorithm in which doctors propose contracts sequentially. The hospitals accumulate the proposed contracts and, at the end of each step, hold their favorite set of contracts among those contracts that have been proposed; in the next step, some doctor for whom no hospital holds a contract proposes his favorite contract that has not yet been proposed. The algorithm ends when no doctor wishes to make an additional proposal; the outcome of the mechanism is comprised of the contracts held at the last step.

Our first main result, Theorem 1, shows why we focus on cumulative offer mechanisms: *whenever a stable and strategy-proof mechanism can be guaranteed to exist, it is equivalent to a cumulative offer mechanism*. Thus, our results can help explain the ubiquity of cumulative offer mechanisms in practice—they are the only candidates for stable and strategy-proof matching mechanisms. Indeed, all of the applications cited here either explicitly use a cumulative offer mechanism or use the deferred acceptance mechanism of Gale and Shapley (1962), which (as we show) is equivalent to a cumulative offer mechanism whenever a stable and strategy-proof mechanism can be guaranteed to exist.

Our Theorem 1 implies that, to characterize when a stable and strategy-proof mechanism can be guaranteed to exist, it is enough to characterize when cumulative offer mechanisms are stable and strategy-proof. En route to such a characterization, we first define an *observable offer process for the hospital  $h$*  as a sequence of contracts with  $h$  in which, for each contract  $x$  in the sequence, the doctor associated with  $x$  is not employed by  $h$  when  $h$  is allowed to choose from all previous contracts in the sequence;<sup>4</sup> thus, an observable offer process corresponds to a sequence of offers that a hospital could receive during a cumulative offer mechanism. We say that the preferences of  $h$  are *observably substitutable* if the set of contracts not chosen by  $h$  weakly expands along any observable sequence of contracts; thus, when the preferences of each hospital are observably substitutable, no hospital chooses a contract it previously rejected during a cumulative offer mechanism. Observable substitutability exactly delineates the two cases of Theorem 1: When the preferences of some hospital are not observably substitutable, it is easy to construct simple preferences for the other hospitals

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<sup>3</sup>From now on, we use the terminology of doctors and hospitals instead of workers and firms, to maintain consistency with the preceding literature.

<sup>4</sup>In other words, an offer process, i.e., a sequence of contracts, is “observable” for  $h$  if it could arise as a sequence of “proposals” from doctors to  $h$  where a doctor only makes a new proposal if all of his old proposals have been rejected.

such that stable and strategy-proof matching is impossible (Theorem 1a). By contrast, when the preferences of each hospital are observably substitutable, any stable and strategy-proof mechanism corresponds to a cumulative offer mechanism (Theorem 1b).<sup>5</sup>

Similarly, we say that the preferences of  $h$  are *observably size monotonic* if the number of contracts chosen by  $h$  weakly increases along any observable sequence of contracts; thus, when the preferences of each hospital are observably size monotonic, each hospital chooses weakly more contracts at each successive step of a cumulative offer mechanism. We show that observable size monotonicity of hospitals' preferences is also necessary for the guaranteed existence of a stable and strategy-proof mechanism (Theorem 2).

However, while observable substitutability and observable size monotonicity are necessary for stable and strategy-proof matching, they are not sufficient. To complete our characterization, we introduce a third condition, which requires that the choice function of hospital  $h$  is *non-manipulable via contractual terms*; that is, if  $h$  is the only hospital, then cumulative offer mechanisms are strategy-proof. Non-manipulability via contractual terms is necessary for the existence of a stable and strategy-proof mechanism because cumulative offer mechanisms are the only candidates for stable and strategy-proof mechanisms when preferences are observably substitutable (Theorem 3).

Non-manipulability via contractual terms completes our characterization: observable substitutability, observable size monotonicity, and non-manipulability via contractual terms together are sufficient for a cumulative offer mechanism to be stable and strategy-proof. In particular, given observable substitutability and observable size monotonicity, strategy-proofness of cumulative offer mechanisms in each single-hospital economy implies that cumulative offer mechanisms are stable and strategy-proof when all hospitals are present. Thus, *a stable and strategy-proof mechanism can be guaranteed to exist if and only if hospitals' preferences are observably substitutable, observably size monotonic, and non-manipulable via contractual terms* (Theorem 4).<sup>6</sup>

In pure matching settings (i.e., matching without contracts), observable substitutability is equivalent to substitutability and observable size monotonicity is equivalent to size monotonicity; hence, prior work (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2008) implies that

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<sup>5</sup>In particular, as we have already remarked, when hospitals' preferences are observably substitutable, the well-known deferred acceptance mechanism is equivalent to the cumulative offer mechanism. The deferred acceptance mechanism proceeds like the cumulative offer mechanism, except that at each step a hospital can only choose a newly proposed contract or a contract that it held previously. Under observable substitutability, no hospital will ever choose a contract previously rejected during a cumulative offer mechanism, and so the outcomes of the deferred acceptance and cumulative offer mechanisms will coincide (see Proposition A.1).

<sup>6</sup>Technically, all of the results described here require an additional technical condition, the *irrelevance of rejected contracts condition*, which requires that an agent's chosen set of contracts does not change when that agent loses access to an unchosen contract (see Aygün and Sönmez (2012, 2013)).

observable substitutability and observable size monotonicity are necessary and sufficient to guarantee the existence of a stable and strategy-proof mechanism in pure matching settings. In some sense, then, observable substitutability and observable size monotonicity rule out a doctor benefitting by misrepresenting his preferences over hospitals. Non-manipulability via contractual terms, by contrast, is vacuously satisfied in pure matching settings, as there is only one possible set of contractual terms between each doctor–hospital pair. However, in settings with contractual terms, non-manipulability via contractual terms is needed to rule out a doctor benefitting by misrepresenting his preferences over contractual terms with a given hospital; this turns out to be exactly the additional requirement necessary to characterize the class of choice functions for which a stable and strategy-proof mechanism can be guaranteed to exist.

Our characterization result relies on the possibility that a doctor can have any preference ordering over contracts involving him (and the outside option); this is natural when contracts encode tasks (such as research or clinical work, or subspecialty assignment) over which doctors have heterogenous preferences. In many real-world settings, however, there are at least some contractual terms over which only certain forms of doctors’ preferences would be reasonable. For example, if doctors prefer higher wages (while holding other contractual terms constant), then it is no longer true that a doctor can have any preference ordering over contracts; rather, a doctor can only have a preference ordering that, given two contracts that coincide but for the wage, ranks the contract with a higher wage above the contract with a lower wage. Similarly, doctors should prefer contracts that are less restrictive to those that are more restrictive: For instance, in the [Sönmez and Switzer \(2013\)](#) setting of cadet–branch matching, some contracts include a requirement for reenlistment; here, it is natural to assume that cadets prefer a contract without a reenlistment obligation. We thus extend our results to the case in which only a subset of all possible rankings is permissible for each doctor: [Theorem 5](#) generalizes our characterization result to show that a stable and strategy-proof mechanism can be guaranteed to exist if and only if our three key properties hold for all offer processes consistent with feasible doctor preferences.

For a simple example of how our results apply in the presence of *a priori* restrictions on doctor preferences, consider the setting of matching with wages ([Kelso and Crawford, 1982](#)), in which the only contractual term is the wage and hospitals’ preferences are quasilinear in the sum of wages paid; in that setting, a stable and strategy-proof mechanism can be guaranteed to exist if and only if hospital choice functions satisfy the classical substitutability and size monotonicity conditions ([Hatfield and Kojima, 2008](#)). The result of [Hatfield and Kojima \(2008\)](#) is a special case of our [Theorem 5](#): With the restriction that doctors prefer higher wages, our conditions reduce to the classical substitutability and size monotonicity conditions

in the [Kelso and Crawford \(1982\)](#) matching with wages setting.

Similarly, our result for settings with restrictions on doctors' preferences generalizes the results of [Abizada and Dur \(2017\)](#).<sup>7</sup> [Abizada and Dur \(2017\)](#) studied the setting of college admissions with financial aid and a budget constraint for each college; in their setting, colleges have responsive preferences over students but are also subject to a budget constraint on the total amount of financial aid available. While such budget constraints induce non-substitutabilities in college preferences, [Abizada and Dur \(2017\)](#) showed that nevertheless a stable and strategy-proof mechanism is guaranteed to exist in their setting. The key to [Abizada and Dur](#)'s result is that students always prefer more financial aid; the requirement that students prefer higher levels of financial aid corresponds in our setting to the requirement that doctors prefer higher wages. Consequently, the [Abizada and Dur \(2017\)](#) result is a special case of our result with restrictions on feasible doctor preferences: In the [Abizada and Dur \(2017\)](#) setting, for all offer processes consistent with this restriction, college choice functions satisfy our three key properties; and so [Theorem 5](#) implies that the cumulative offer mechanism is stable and strategy-proof.<sup>8</sup> Moreover, [Theorem 5](#) allows us to identify the complete space of college choice functions for which a stable and strategy-proof mechanism can be guaranteed to exist when students are known to prefer more financial aid (and there are possibly other contractual terms, such as field of study).

In the final part of our paper, we characterize the class of choice functions for which cumulative offer mechanisms are guaranteed to yield stable outcomes. We say that the preferences of a hospital  $h$  are *observably substitutable across doctors* if  $h$  never chooses a previously-rejected contract with a doctor not currently employed by  $h$  along any observable offer process. We show that if the preferences of each hospital are observably substitutable across doctors, then the outcome of a cumulative offer mechanism is independent of proposal order ([Proposition 3](#)). Moreover, cumulative offer mechanisms are guaranteed to produce stable outcomes ([Theorem 6](#)). By contrast, if the preferences of any hospital are not observably substitutable across doctors, then there exist unit-demand preferences for the other hospitals such that no cumulative offer mechanism is stable ([Theorem 7](#)). However, we demonstrate by means of an example that there exists a larger class of hospital preferences for which stable outcomes are guaranteed to exist. Hence, if one is only interested in achieving stable outcomes and does not care about incentive compatibility, it is not sufficient to restrict attention to cumulative offer mechanisms.

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<sup>7</sup>[Abizada and Dur \(2017\)](#) built on previous work by [Abizada \(2016\)](#).

<sup>8</sup>Building on our work here, [Schlegel \(2019\)](#) made this connection precise and uses our [Theorem 5](#) to identify the complete space of college choice functions for which a stable and strategy-proof mechanism can be guaranteed to exist when the only contractual term is financial aid and students are known to prefer more financial aid.

## 1.1 Related Literature

The numerous real-world applications of matching under non-substitutable preferences have motivated work to find conditions on firms’ preferences weaker than substitutability and size monotonicity that still guarantee the existence of stable and strategy-proof mechanisms. Progressively weaker conditions have been found for which stable and strategy-proof matching can be achieved; however, under all of these conditions the same mechanism has been found to be stable and strategy-proof—the cumulative offer mechanism.

Our work implicitly explains why the cumulative offer mechanism has been so central in prior work by showing that whenever a stable and strategy-proof matching can be guaranteed to exist, the cumulative offer mechanism is the unique such mechanism. At the same time, we unify the previous work on sufficient conditions for stable and strategy-proof matching: [Hatfield and Kojima \(2010\)](#) introduced a weakened substitutability condition called *unilateral substitutability* and showed that when all firms’ preferences are unilaterally substitutable (and size monotonic), the cumulative offer mechanism is stable and strategy-proof. [Kominers and Sönmez \(2016\)](#) identified a novel class of preferences, called *slot-specific priorities*, and showed that if all firms’ preferences are in this class, then the cumulative offer mechanism is stable and strategy-proof. [Hatfield and Kominers \(2019\)](#) developed a concept of *substitutable completion* and showed that when each firm’s preferences admit a size monotonic substitutable completion, the cumulative offer mechanism is stable and strategy-proof.<sup>9</sup> As our work fully characterizes the necessary and sufficient conditions for stable and strategy-proof matching, it necessarily subsumes all of the prior conditions; see [Section 3.5](#) for details.<sup>10</sup>

Earlier work by [Alcalde and Barberà \(1994\)](#) in the setting of many-to-one matching (without contracts) identified the cumulative offer mechanism as the unique individually rational, non-wasteful, fair, and strategy-proof mechanism when hospital preferences are responsive in the sense of [Roth \(1982\)](#); thus, the cumulative offer mechanism is the unique stable and strategy-proof mechanism in their setting.<sup>11,12</sup> [Balinski and Sönmez \(1999\)](#) then

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<sup>9</sup>[Zhang \(2016\)](#) and [Kadam \(2017\)](#) showed that all unilaterally substitutable preferences are substitutably completable; [Hatfield and Kominers \(2019\)](#) showed that the slot-specific priorities of [Kominers and Sönmez \(2016\)](#) always admit a size monotonic substitutable completion.

<sup>10</sup>In a companion paper ([Hatfield et al., 2017](#)) we apply the results here to show that settings with firms that have certain cross-division hiring restrictions admit stable and strategy-proof matching.

<sup>11</sup>[Alcalde and Barberà \(1994\)](#) actually showed that the deferred acceptance mechanism is the unique stable and strategy-proof mechanism in their setting, but the deferred acceptance and cumulative offer mechanisms are equivalent when hospitals’ preferences are responsive.

<sup>12</sup>[Alcalde and Barberà \(1994\)](#) also showed that the cumulative offer mechanism is the unique mechanism that is stable and strategy-proof for workers *and* firms provided that firms’ preferences over groups of workers are responsive and firms’ preferences over individual workers additionally satisfy the “top dominance” condition.

showed that deferred acceptance is the unique stable mechanism that respects improvements in the setting of many-to-one matching with responsive preferences for hospitals. Our analysis differs from that in [Alcalde and Barberà \(1994\)](#) as we consider the setting of matching-with-contracts and do not restrict hospitals’ preferences to be responsive.

Meanwhile, [Hirata and Kasuya \(2017\)](#) have shown in the setting of many-to-one matching with contracts that there exists at most one stable and strategy-proof mechanism for any fixed profile of choice functions that satisfy the irrelevance of rejected contracts condition.<sup>13</sup> However, [Hirata and Kasuya \(2017\)](#) do not provide any characterization of conditions under which a stable and strategy-proof mechanism is guaranteed to exist, nor do they characterize the class of mechanisms that *could* be stable and strategy-proof. By contrast, our methods allow us to completely characterize the class of stable and strategy-proof mechanisms for any choice function domains that include all unit-demand choice functions. Furthermore, we provide exact conditions on choice functions under which a stable and strategy-proof mechanism is guaranteed to exist (Theorem 4).

## 1.2 Organization of the Paper

The remainder of the paper is organized as follows: Section 2 introduces the many-to-one matching with contracts framework. Section 3 proves our characterization results for stable and strategy-proof mechanisms. Section 4 provides conditions under which the cumulative offer mechanism always produces a stable outcome. Section 5 concludes. We discuss deferred acceptance mechanisms in Appendix A; most of the proofs are presented in Appendix B. A separate Online Appendix contains additional examples and discussion, as well as the proof of an important auxiliary result.

# 2 Model

## 2.1 Framework

There is a finite set of *doctors*  $D$  and a finite set of *hospitals*  $H$ . There is also a finite set of *contracts*  $X$ , with each  $x \in X$  identified with a unique doctor  $\mathbf{d}(x)$  and a unique hospital  $\mathbf{h}(x)$ ; there may be many contracts between the same doctor–hospital pair. To simplify the statements of our results, we assume throughout that for each hospital  $h$  and each doctor  $d$  there exists at least one contract  $x$  such that  $\mathbf{d}(x) = d$  and  $\mathbf{h}(x) = h$ . An *outcome* is a set

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<sup>13</sup>We generalize this result of [Hirata and Kasuya \(2017\)](#) in our Proposition 1, where we show that, for any given profile of choice functions, there can be at most one mechanism that is stable and that satisfies the weaker incentive compatibility requirement of *truncation-consistency*.



of contracts  $Y \subseteq X$ . For an outcome  $Y$ , we let  $\mathbf{d}(Y) \equiv \cup_{y \in Y} \{\mathbf{d}(y)\}$  and  $\mathbf{h}(Y) \equiv \cup_{y \in Y} \{\mathbf{h}(y)\}$ . For any  $i \in D \cup H$ , we let  $Y_i \equiv \{y \in Y : i \in \{\mathbf{d}(y), \mathbf{h}(y)\}\}$ . An outcome  $Y \subseteq X$  is *feasible* if for all  $d \in D$ ,  $|Y_d| \leq 1$ . We say that  $d$  is *unemployed under*  $Y$  if  $Y_d = \emptyset$ .

Each hospital  $h \in H$  has multi-unit demand over contracts in  $X_h$  and is endowed with a choice function  $C^h$  that describes the hospital's choice from any available set of contracts, i.e.,  $C^h(Y) \subseteq Y$  for all  $Y \subseteq X$ . We assume throughout that for all  $Y \subseteq X$ , each hospital  $h \in H$

- (1) only chooses contracts to which it is a party, i.e.,  $C^h(Y) \subseteq Y_h$ ,
- (2) signs at most one contract with any given doctor, i.e.,  $C^h(Y)$  is feasible, and
- (3) considers rejected contracts to be irrelevant, i.e., for all  $x \in X$ , if  $x \notin C^h(\{x\} \cup Y)$ , then  $C^h(\{x\} \cup Y) = C^h(Y)$ .<sup>14</sup>

A class of particularly simple choice functions for hospitals is the class of unit-demand choice functions; a hospital  $h$  has *unit demand* if  $|C^h(Y)| \leq 1$  for all  $Y \subseteq X$ .

In our examples, it will be helpful to describe choice functions as deriving from strict preference rankings over sets of contracts. A strict preference relation  $\succ_h$  for hospital  $h$  over subsets of  $X_h$  *induces* a choice function  $C^h$  for  $h$  under which

$$C^h(Y) = \max_{\succ_h} \{Z \subseteq X_h : Z \subseteq Y\},$$

where by  $\max_{\succ_h}$  we mean the maximum with respect to the ordering  $\succ_h$ ; that is,  $h$  chooses its most-preferred subset of  $Y$ .<sup>15</sup>

We denote by  $C^H(Y) \equiv \cup_{h \in H} C^h(Y)$  the set of contracts chosen by the set of all hospitals from a set of contracts  $Y \subseteq X$ . For any  $Y \subseteq X$  and  $h \in H$ ,  $R^h(Y) \equiv Y_h \setminus C^h(Y)$  denotes the set of contracts that  $h$  rejects from  $Y$ .<sup>16</sup>

Each doctor  $d \in D$  has *unit demand* over contracts in  $X_d$  and an *outside option*  $\emptyset$ . We denote the strict preferences of doctor  $d$  over  $X_d \cup \{\emptyset\}$  by  $\succ_d$ . A contract  $x \in X_d$  is *acceptable (with respect to  $\succ_d$ )* if  $x \succ_d \emptyset$ . We extend the specification of doctor preferences over contracts to preferences over outcomes in the natural way.<sup>17</sup>

<sup>14</sup>The importance of this *irrelevance of rejected contracts* condition is discussed by [Aygün and Sönmez \(2012, 2013\)](#).

<sup>15</sup>Note that a choice function induced by a preference relation automatically satisfies the irrelevance of rejected contracts condition.

<sup>16</sup>Note that this definition of  $R^h(Y)$  is somewhat non-standard, as under this definition,  $R^h(Y)$  does *not* contain the contracts in  $Y$  that are not associated with  $h$ .

<sup>17</sup>That is, for each doctor  $d \in D$ :

1. for any outcome  $Y$  such that  $|Y_d| > 1$ , we let  $\emptyset \succ_d Y$ ;
2. for any outcome  $Y$  such that  $Y_d = \emptyset$ , we let  $Y \sim_d \emptyset$ ;

Much of our analysis relies on examining how the choice of a hospital changes over the course of a sequence of contractual offers. Consider an arbitrary hospital  $h \in H$  whose choice function is given by  $C^h$ . An *offer process* for  $h$  is a finite sequence of distinct contracts  $(x^1, \dots, x^M)$  such that, for all  $m = 1, \dots, M$ , we have that  $x^m \in X_h$ . An offer process  $(x^1, \dots, x^M)$  for  $h$  is *observable* if, for all  $m = 1, \dots, M$ , we have that  $d(x^m) \notin d(C^h(\{x^1, \dots, x^{m-1}\}))$ . Intuitively, an observable offer process for hospital  $h$  is a sequence of contract offers proposed by doctors with the constraint that a doctor can propose  $x^m$  only if no contract with that doctor is chosen by  $h$  when  $h$  has access to  $\{x^1, \dots, x^{m-1}\}$ . Given an observable offer process  $\mathbf{x} = (x^1, \dots, x^M)$  for  $h$ , we denote by  $\mathbf{c}(\mathbf{x}) = \{x^1, \dots, x^M\}$  the set of all contracts in the offer process.

Finally, it will be helpful to compare our results to those in *pure matching settings*, where for each doctor–hospital pair  $(d, h)$  we have that  $|X_d \cap X_h| = 1$ , i.e., there is at exactly one contract between each doctor and hospital.

## 2.2 Stability

We now define the standard solution concept for matching with contracts ([Hatfield and Milgrom, 2005](#)).

**Definition 1.** An outcome  $A \subseteq X$  is *stable* if it is:

1. *Individually rational:*  $C^H(A) = A$  and, for all  $d \in D$ ,  $A_d \succeq_d \emptyset$ .
2. *Unblocked:* There does not exist a nonempty  $Z \subseteq (X \setminus A)$  such that  $Z \subseteq C^H(A \cup Z)$  and, for all  $d \in d(Z)$ ,  $Z \succ_d A$ .

Our definition of stability requires that no agent wishes to unilaterally drop a contract and that there does not exist a *blocking set*  $Z$  such that all hospitals and doctors associated with contracts in  $Z$  want to sign all of the contracts in  $Z$ —potentially after dropping some of the contracts in  $A$ .

## 2.3 Substitutability and Size Monotonicity

A choice function  $C^h$  is *substitutable* if no two contracts  $x$  and  $z$  are “complements” under  $C^h$ , in the sense that having access to  $x$  makes  $z$  more attractive. That is,  $C^h$  is *substitutable* if for

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3. for any two outcomes  $Y$  and  $Z$  such that  $Y_d = \{y\}$  and  $Z_d = \{z\}$ , we let  $Y \succ_d Z$  if and only if  $y \succ_d z$ ;
  4. for any two outcomes  $Y$  and  $Z$  such that  $Y_d = \{y\}$  and  $Z_d = \emptyset$ , we let  $Y \succ_d Z$  if and only if  $y \succ_d \emptyset$ ;  
and
  5. for any two outcomes  $Y$  and  $Z$  such that  $Y_d = \emptyset$  and  $Z_d = \{z\}$ , we let  $Y \succ_d Z$  if and only if  $\emptyset \succ_d z$ .

all contracts  $x$  and  $z$  and sets of contracts  $Y$ , if  $z \notin C^h(Y \cup \{z\})$ , then  $z \notin C^h(\{x\} \cup Y \cup \{z\})$ .<sup>18</sup>

Substitutability is equivalent to monotonicity of the rejection function:  $C^h$  is substitutable if and only if we have  $R^h(Y) \subseteq R^h(Z)$  for all sets of contracts  $Y$  and  $Z$  such that  $Y \subseteq Z$ . The choice function  $C^h$  is *size monotonic* if  $h$  chooses weakly more contracts whenever the set of available contracts expands, i.e., if for all contracts  $z$  and sets of contracts  $Y$ , we have  $|C^h(Y)| \leq |C^h(Y \cup \{z\})|$ .<sup>19</sup>

## 2.4 Mechanisms

Given a profile of choice functions  $C = (C^h)_{h \in H}$ , a *mechanism*  $\mathcal{M}(\cdot; C)$  maps preference profiles for the doctors  $\succ = (\succ_d)_{d \in D}$  to outcomes. Most of the time, we shall assume that the choice functions of the hospitals are fixed and write  $\mathcal{M}(\succ)$  in place of  $\mathcal{M}(\succ; C)$ . For future reference, we set  $\mathcal{M}_d(\succ) \equiv [\mathcal{M}(\succ)]_d = \mathcal{M}(\succ) \cap X_d$  for all  $d \in D$  and  $\mathcal{M}_h(\succ) \equiv [\mathcal{M}(\succ)]_h = \mathcal{M}(\succ) \cap X_h$  for all  $h \in H$ . We will also occasionally abuse notation and write, for a doctor  $d$ ,  $\mathcal{M}_d(\succ) = x$  instead of  $\mathcal{M}_d(\succ) = \{x\}$ , in either case denoting that  $d$  received the contract  $x$ . Similarly, we will sometimes write  $\mathcal{M}_d(\succ) = \emptyset$  instead of  $\mathcal{M}_d(\succ) = \emptyset$ . We say that two mechanisms  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  are *outcome-equivalent* if  $\mathcal{M}(\succ) = \bar{\mathcal{M}}(\succ)$  for all preference profiles  $\succ$ .

A mechanism  $\mathcal{M}$  is *stable* if  $\mathcal{M}(\succ)$  is a stable outcome for every preference profile  $\succ$ . A mechanism  $\mathcal{M}$  is *strategy-proof* if for every preference profile  $\succ$ , and for each doctor  $d \in D$ , there does not exist a  $\hat{\succ}_d$  such that  $\mathcal{M}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{M}(\succ)$ . It will prove useful to define a weaker notion of incentive compatibility that only requires mechanisms to be immune to misreports that change the rank of the outside option. Formally, for any  $d \in D$ , we say that  $\hat{\succ}_d$  is a *truncation* of  $\succ_d$  if

- for all  $y, z \in X_d$ ,  $y \hat{\succ}_d z$  if and only if  $y \succ_d z$ , and
- for all  $y \in X_d$ ,  $y \hat{\succ}_d \emptyset$  only if  $y \succ_d \emptyset$ .

A mechanism  $\mathcal{M}$  is *truncation-consistent* if, for any doctor  $d \in D$ , any contract  $x \in X_d$ , any preference profile  $\succ$ , and any preference relation  $\hat{\succ}_d$  such that  $\hat{\succ}_d$  is a truncation of  $\succ_d$  and  $x \hat{\succ}_d \emptyset$ , we have that  $\mathcal{M}_d(\succ) = x$  if and only if  $\mathcal{M}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = x$ . Any strategy-proof mechanism must be truncation-consistent, as a truncation  $\hat{\succ}_d$  is a special case of the types of manipulations ruled out by strategy-proofness.

One class of mechanisms of particular importance is the class of *cumulative offer mechanisms*. A cumulative offer mechanism is defined with respect to a strict ordering  $\vdash$  of the

<sup>18</sup>The substitutability condition was introduced by [Kelso and Crawford \(1982\)](#) and adapted to settings with limited transfers by [Roth \(1984\)](#).

<sup>19</sup>Size monotonicity is called the *Law of Aggregate Demand* by [Hatfield and Milgrom \(2005\)](#).

elements of  $X$ . For any preference profile  $\succ$  and ordering  $\vdash$ , the outcome of the cumulative offer mechanism, denoted by  $C^\vdash(\succ)$ , is determined by the *cumulative offer process with respect to  $\vdash$  and  $\succ$*  as follows:

**Step 0:** Initialize the set of contracts *available* to the hospitals as  $A^0 = \emptyset$ .

**Step  $t \geq 1$ :** Consider the set

$$U^t \equiv \{x \in X \setminus A^{t-1} : \mathbf{d}(x) \notin \mathbf{d}(C^H(A^{t-1})) \text{ and } \nexists z \in (X_{\mathbf{d}(x)} \setminus A^{t-1}) \cup \{\emptyset\} \text{ such that } z \succ_{\mathbf{d}(x)} x\}.$$

If  $U^t$  is empty, then the algorithm terminates and the outcome is given by  $C^H(A^{t-1})$ . Otherwise, letting  $y^t$  be the highest-ranked element of  $U^t$  according to  $\vdash$ , we say that  $y^t$  is *proposed* and set  $A^t = A^{t-1} \cup \{y^t\}$  and proceed to step  $t + 1$ .

A cumulative offer process begins with no contracts available to the hospitals (i.e.,  $A^0 = \emptyset$ ). Then, at each step  $t$ , we construct  $U^t$ , the set of contracts that (1) have not yet been proposed, (2) are not associated to doctors associated with contracts chosen by hospitals from the currently available set of contracts, and (3) are both acceptable and the most-preferred by their associated doctors among all contracts not yet proposed. If  $U^t$  is empty, then every doctor  $d$  either has some associated contract chosen by some hospital, i.e.,  $d \in \mathbf{d}(C^H(A^{t-1}))$ , or has no acceptable contracts left to propose, and so the cumulative offer process ends. Otherwise, the contract in  $U^t$  that is highest-ranked according to  $\vdash$  is proposed by its associated doctor, and the process proceeds to the next step. Note that at some step this process must end as the number of contracts is finite.

Letting  $T$  denote the last step of the cumulative offer process with respect to  $\vdash$  and  $\succ$ , we call  $A^T$  the set of contracts *observed* in the cumulative offer process with respect to  $\vdash$  and  $\succ$ . Note that without further assumptions on hospitals' choice functions, the outcome of a cumulative offer mechanism need not be feasible, i.e., it might be the case that  $C^H(A^T)$  contains more than one contract with a given doctor.<sup>20</sup>

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<sup>20</sup>Recall that no individual hospital ever chooses more than one contract with a given doctor. Hence, if  $C^H(A^T)$  is infeasible, then there must exist at least one doctor who is employed by two distinct hospitals at the set of accumulated offers  $A^T$ .

## 2.5 Guaranteeing the Existence of Stable and Strategy-Proof Mechanisms

A class  $\mathcal{C}^h$  for hospital  $h$  is a subset of the set of all possible choice functions for hospital  $h$ . We say that a class  $\mathcal{C}^h$  is *unital* if it includes all unit-demand choice functions for  $h$ .<sup>21,22</sup> A profile of classes  $\mathcal{C} \equiv \times_{h \in H} \mathcal{C}^h$  is *unital* if  $\mathcal{C}^h$  is unital for each  $h \in H$ .

A mechanism satisfying certain properties is *guaranteed to exist* (for a profile of classes  $\mathcal{C}$ ) if, whenever  $C = (C^h)_{h \in H}$  is such that  $C^h \in \mathcal{C}^h$  for each  $h \in H$ , a mechanism  $\mathcal{M}(\cdot; C)$  satisfying those properties exists.<sup>23</sup> Our main goal is to characterize the maximal unital profile of classes for which a stable and strategy-proof mechanism is guaranteed to exist.<sup>24</sup> That is, we wish to find the most general conditions on hospitals' choice functions that include every unit-demand choice function and—when imposed separately on the choice function of each hospital—guarantee the existence of a stable and strategy-proof mechanism.

The restriction to profiles of classes that contain all unit-demand choice functions is standard in matching theory: In practice, it is natural to assume that hospitals might have unit-demand preferences, i.e., have only one open position, and that there are no *a priori* restrictions on how the hospital ranks different contracts/doctors. Any domain of profiles of choice functions under which a stable and strategy-proof mechanism is guaranteed to exist and that is *not* fully contained in the maximal unital profile of classes that we characterize here must either rule out some unit-demand choice functions or require some form of interdependence across hospitals' choice functions.<sup>25</sup>

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<sup>21</sup>Recall that, for  $C^h$  to be a valid choice function for hospital  $h$ , it must be the case that  $h$  only chooses contracts to which  $h$  is a party,  $h$  chooses at most one contract with any doctor, and  $h$ 's choice does not change if a rejected contract becomes unavailable.

<sup>22</sup>For our proofs, it will actually be sufficient to assume that, for each hospital  $h$ ,  $\mathcal{C}^h$  is large enough to allow for arbitrary choices from sets of contracts that contain no more than one contract per doctor. More precisely, we need that, for any three contracts  $x, y, z \in X_h$  such that  $|\{\mathbf{d}(x), \mathbf{d}(y), \mathbf{d}(z)\}| = 3$ , there exists a (unit-demand) choice function  $C^h \in \mathcal{C}^h$  such that  $C^h(\{x, y, z\}) = \{x\}$ ,  $C^h(\{y, z\}) = \{y\}$ , and  $C^h(\{z\}) = \{z\}$ .

<sup>23</sup>For instance, if  $\mathcal{C}^h$  is the set of substitutable and size monotonic choice functions for hospital  $h$ , the results of [Hatfield and Milgrom \(2005\)](#) imply that a stable and strategy-proof mechanism is guaranteed to exist for  $\mathcal{C}$ .

<sup>24</sup>In particular, our results show that there is a unique profile of classes that assures existence and is maximal among all unital profiles of classes.

<sup>25</sup>Ruling out unit-demand choice functions seems highly problematic, as we discussed at the beginning of this paragraph. Moreover, in most applications, there are no natural interdependencies across hospitals' choice functions.

For the setting with interdependencies across hospitals' choice functions, [Pycia \(2012\)](#) established a maximal domain result for the existence of stable outcomes in a class of coalition formation problems that includes many-to-one matching problems (without contracts). However, the characterization of [Pycia \(2012\)](#) implicitly relies on the existence of peer effects, that is, on the assumption that doctors care about more than just the hospitals they are assigned to. If there are no peer effects, the key preference restriction developed in [Pycia \(2012\)](#), *pairwise alignment*, is not necessary for the existence of stable outcomes, and also unlikely to be satisfied.

### 3 Stable and Strategy-Proof Mechanisms

As discussed in the Introduction, cumulative offer mechanisms have been shown to be stable and strategy-proof in a wide range of many-to-one matching settings. But, in principle, stable and strategy-proof matching could require that the form of the mechanism depend on the types of choice functions that hospitals may have. In particular, it could be the case that for some classes of choice functions, stable and strategy-proof mechanisms are guaranteed to exist even though no cumulative offer mechanism is stable and strategy-proof. Our first main result shows that, in fact, this is *not* the case: Whenever a stable and strategy-proof mechanism is guaranteed to exist, *any* stable and strategy-proof mechanism is equivalent to a cumulative offer mechanism.

**Theorem 1** (FOUNDATION FOR CUMULATIVE OFFER MECHANISMS). *Let  $|H| > 1$ , let  $\vdash$  be some ordering over  $X$ , and fix a profile of choice functions  $C$ . Then either*

- *there exists  $h \in H$ , along with unit-demand choice functions  $\bar{C}^{H \setminus \{h\}}$  for the other hospitals, such that no stable and strategy-proof mechanism exists for  $(C^h, \bar{C}^{H \setminus \{h\}})$ , or*
- *every stable and strategy-proof mechanism for  $C$  is outcome-equivalent to  $C^+$ .*

Our first main result, Theorem 1, provides the foundation for the central role of cumulative offer mechanisms in theory and helps explain why these mechanisms are prevalent in practice. Indeed, cumulative offer mechanisms are essentially the *only* candidates for stable and strategy-proof matching: Whenever a stable and strategy-proof mechanism is guaranteed to exist, every mechanism that is stable and strategy-proof is outcome-equivalent to a cumulative offer mechanism.<sup>26</sup> As a consequence, when considering whether a stable and strategy-proof mechanism exists, we need only consider cumulative offer mechanisms. Note that our first main result—like most of our other results—relies on the assumption that hospitals can have any unit-demand choice function. By contrast, when we fix the entire profile of choice functions, a stable and strategy-proof mechanism may exist and, moreover, that mechanism may not be outcome-equivalent to any cumulative offer mechanism; see Example 3 of [Hirata and Kasuya \(2017\)](#) for such an example. However, Theorem 1 does imply that any mechanism that is not outcome-equivalent to a cumulative offer mechanism can only be stable and strategy-proof when we rule out some unit-demand choice functions or assume some form of interdependency across hospitals' choice functions.

In order to develop intuition for Theorem 1, note first that when searching for a stable and strategy-proof mechanism, cumulative offer mechanisms are natural candidates to consider

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<sup>26</sup>In particular, whenever a stable and strategy-proof mechanism is guaranteed to exist, all cumulative offer mechanisms are outcome-equivalent; see Theorem 1b.

as they *always* (i.e., for any choice functions of the hospitals) produce unblocked outcomes: At each step of a cumulative offer mechanism, some hospital gains access to a new contract; thus, at the last step, each doctor has already offered every contract that he prefers to the final contract he offered. Hence, every hospital already has “access to” all of the contracts that could be used to comprise blocking sets at the end of a cumulative offer mechanism, and since each hospital then chooses its favorite set of contracts, no blocking set can exist. Moreover, it is immediate that each hospital has an individually rational set of contracts, because each hospital chooses its favorite subset from the set of contracts offered to it.

Thus, cumulative offer mechanisms always produce outcomes that are unblocked and individually rational for hospitals. However, a cumulative offer mechanism may fail to produce an outcome that is individually rational for doctors. In particular, at the end of a cumulative offer mechanism, two different hospitals may hold a contract with the same doctor, as we show in the next example. But when a cumulative offer mechanism assigns two contracts to the same doctor, we find ourselves in the first case of Theorem 1.

**Example 1.** Suppose that there are two hospitals, two doctors, and a contract between each doctor and each hospital. More formally, let  $H = \{h, \hat{h}\}$ ,  $D = \{d, e\}$ , and  $X = \{x, \hat{x}, y, \hat{y}\}$ , where  $d = \mathbf{d}(x) = \mathbf{d}(\hat{x})$ ,  $e = \mathbf{d}(y) = \mathbf{d}(\hat{y})$ ,  $h = \mathbf{h}(x) = \mathbf{h}(y)$ , and  $\hat{h} = \mathbf{h}(\hat{x}) = \mathbf{h}(\hat{y})$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\{x, y\} \succ \{y\} \succ \emptyset$$

and the choice function  $C^{\hat{h}}$  of  $\hat{h}$  be induced by the preference relation

$$\{\hat{x}\} \succ \{\hat{y}\} \succ \emptyset.$$

Meanwhile, let the preferences of the doctors be given by

$$\begin{aligned} \succ_d &: x \succ \hat{x} \succ \emptyset \\ \succ_e &: \hat{y} \succ y \succ \emptyset. \end{aligned}$$

Consider the cumulative offer mechanism  $\mathcal{C}^+$  with ordering  $x \vdash \hat{x} \vdash y \vdash \hat{y}$ . In the first step of  $\mathcal{C}^+$ , we have that  $U^1 = \{x, \hat{y}\}$  and so  $A^1 = \{x\}$  and  $C^H(A^1) = \emptyset$ . In the second step,  $U^2 = \{\hat{x}, \hat{y}\}$  and so  $A^2 = \{x, \hat{x}\}$  and  $C^H(A^2) = \{\hat{x}\}$ . In the third step,  $U^3 = \{y\}$  and so  $A^3 = \{x, \hat{x}, \hat{y}\}$  and  $C^H(A^3) = \{\hat{x}\}$ . In the fourth step,  $U^4 = \{y\}$  and so  $A^4 = \{x, \hat{x}, \hat{y}, y\}$  and  $C^H(A^4) = \{\hat{x}, x, y\}$ . Finally, in the fifth step,  $U^5$  is empty and so the algorithm terminates; the outcome is  $\{\hat{x}, x, y\}$ .

While  $\mathcal{C}^+$  fails to produce a stable outcome in Example 1, this is not a failure just of the  $\mathcal{C}^+$  mechanism—no stable outcome exists in the setting of Example 1. To see this, consider any individually rational outcome  $A$ :

- If  $A_{\hat{h}} = \emptyset$ , then  $\{\hat{y}\}$  is a blocking set.
- If  $A_{\hat{h}} = \{\hat{y}\}$ , then  $\{\hat{x}\}$  is a blocking set as  $\{x\}$  is not individually rational for  $h$ .
- If  $A_{\hat{h}} = \{\hat{x}\}$ , then either  $\{x, y\}$  is a blocking set (when  $A_h = \emptyset$ ) or  $\{x\}$  is a blocking set (when  $A_h = \{y\}$ ).

Thus, while no cumulative offer mechanism produces a stable outcome in Example 1, in fact no mechanism can produce a stable outcome.

Example 1 illustrates that cumulative offer mechanisms may fail to produce stable outcomes; however, even when a cumulative offer mechanism is guaranteed to produce stable outcomes, it may fail to be strategy-proof. For instance, when there is only one hospital, cumulative offer mechanisms are always stable (Hatfield and Milgrom, 2005, Theorem 12) but they may not be strategy-proof. In particular, as our next example illustrates, an agent may find it beneficial to “reverse-truncate,” i.e., to report that an unacceptable contract is acceptable. But, when a cumulative offer mechanism fails to be strategy-proof in this way, we again find ourselves in the first case of Theorem 1, and so no mechanism is both stable and strategy-proof.

**Example 2.** Suppose that there is one hospital, two doctors, and two contracts between each doctor and the hospital. More formally, let  $H = \{h\}$ ,  $D = \{d, e\}$ , and  $X = \{x, \hat{x}, y, \hat{y}\}$  where  $d = \mathbf{d}(x) = \mathbf{d}(\hat{x})$ ,  $e = \mathbf{d}(y) = \mathbf{d}(\hat{y})$ ,  $h = \mathbf{h}(x) = \mathbf{h}(y) = \mathbf{h}(\hat{x}) = \mathbf{h}(\hat{y})$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\{x, y\} \succ \{\hat{x}\} \succ \{\hat{y}\} \succ \{x\} \succ \{y\} \succ \emptyset.$$

Meanwhile, let the preferences of the doctors be given by

$$\begin{aligned} \succ_d &: x \succ \emptyset \\ \succ_e &: \hat{y} \succ y \succ \emptyset. \end{aligned}$$

Consider the cumulative offer mechanism  $\mathcal{C}^+$  with ordering  $x \vdash \hat{x} \vdash y \vdash \hat{y}$ . In the first step of  $\mathcal{C}^+$ , we have that  $U^1 = \{x, \hat{y}\}$  and so  $A^1 = \{x\}$  and  $C^H(A^1) = \{x\}$ . In the second step,  $U^2 = \{\hat{y}\}$  and so  $A^2 = \{x, \hat{y}\}$  and  $C^H(A^2) = \{\hat{y}\}$ . In the third step,  $U^3$  is empty and so the algorithm terminates; the final outcome is  $\{\hat{y}\}$ .



However,  $\mathcal{C}^+$  is not strategy-proof: Consider the case in which  $d$ 's preferences are given by

$$\hat{\succ}_d : x \succ \hat{x} \succ \emptyset.$$

For these preferences, the first two steps of the algorithm are identical to those when  $d$ 's preferences are  $x \succ_d \emptyset$ . But in the third step,  $U^3 = \{\hat{x}\}$ , and so  $A^3 = \{x, \hat{y}, \hat{x}\}$  and  $C^H(A^3) = \{\hat{x}\}$ . In the fourth step,  $U^4 = \{y\}$ , and so  $A^4 = \{x, \hat{y}, \hat{x}, y\}$  and  $C^H(A^4) = \{x, y\}$ . Finally, in the fifth step,  $U^5$  is empty and so the algorithm terminates; the outcome is  $\{x, y\}$ .

While cumulative offer mechanisms are not strategy-proof in Example 2, *no* stable mechanism is strategy-proof: Any stable mechanism  $\mathcal{M}$  must choose either  $\{x, y\}$  or  $\{\hat{y}\}$  under the preferences  $\succ$ , as these are the only stable outcomes. If  $\mathcal{M}(\succ) = \{\hat{y}\}$ , then note that under  $(\hat{\succ}_d, \succ_e)$  the only stable outcome is  $\{x, y\}$ ; thus  $\mathcal{M}(\hat{\succ}_d, \succ_e) = \{x, y\}$  and so  $\mathcal{M}$  is not strategy-proof for  $d$ , just as  $\mathcal{C}^+$  was not strategy-proof for  $d$ . But if  $\mathcal{M}(\succ) = \{x, y\}$ , then note that under  $(\succ_d, \hat{\succ}_e)$ , where

$$\hat{\succ}_e : \hat{y} \succ \emptyset,$$

the only stable outcome is  $\{\hat{y}\}$ ; thus  $\mathcal{M}(\succ_d, \hat{\succ}_e) = \{\hat{y}\}$  and so  $\mathcal{M}$  is not strategy-proof for  $e$ . Thus, while no cumulative offer mechanism in Example 2 is strategy-proof, in fact no stable mechanism is strategy-proof.

In the next subsection, we develop a new restriction on hospitals' choice functions that rules out the problems encountered by cumulative offer mechanisms that we found in Examples 1 and 2. We show that our condition is, in a maximal domain sense, necessary for the existence of a stable and strategy-proof mechanism, and that any stable and strategy-proof mechanism has to be equivalent to a cumulative offer mechanism when our condition is satisfied.

### 3.1 Observable Substitutability

The key reason that cumulative offer mechanisms fail to be stable and strategy-proof in both Examples 1 and 2 is that, at some step, the hospital  $h$  chooses a contract that it rejected at some previous step.

In Example 1, the outcome of  $\mathcal{C}^+$  is not individually rational for doctors as the hospitals hold two contracts— $x$  and  $\hat{x}$ —with the doctor  $d$  at the end of  $\mathcal{C}^+$ . The cumulative offer mechanism  $\mathcal{C}^+$  fails here because the hospital  $h$  chooses the contract  $x$  in Step 4 even though  $h$  rejected  $x$  at an earlier step.

In Example 2,  $\mathcal{C}^+$  is not strategy-proof:  $d$  is better off (under  $\succ_d$ ) when the cumulative offer mechanism acts as if  $d$ 's preferences were  $x \hat{\succ}_d \hat{x} \hat{\succ}_d \emptyset$ . The cumulative offer mechanism

$\mathcal{C}^+$  fails because hospital  $h$  chooses the contract  $x$  in Step 4 (under  $(\hat{\succ}_d, \succ_e)$ ) even though  $h$  rejected  $x$  at an earlier step.

We now introduce a condition on choice functions which guarantees that hospitals never want to “take back” a previously rejected contract over the course of a cumulative offer mechanism.

**Definition 2.** A choice function  $C^h$  exhibits an observable violation of substitutability if there exists an observable offer process  $(x^1, \dots, x^M)$  for  $h$  such that  $R^h(\{x^1, \dots, x^{M-1}\}) \setminus R^h(\{x^1, \dots, x^M\}) \neq \emptyset$ . A choice function  $C^h$  is *observably substitutable* if it does not exhibit an observable violation of substitutability.

Note that if all choice functions are observably substitutable, then no hospital will ever want to “take back” a contract it previously rejected during a cumulative offer mechanism: For each individual hospital  $h$ , the sequence of contracts offered to  $h$  during a cumulative offer mechanism is observable; hence, the set of contracts that  $h$  rejects is guaranteed to expand monotonically over the course of any cumulative offer mechanism. Put differently, observable substitutability guarantees that cumulative offer mechanisms proceed as if choice functions are substitutable. In particular, when choice functions are observably substitutable, any cumulative offer mechanism produces a stable outcome (see Theorem 6). Moreover, when choice functions are observably substitutable, every cumulative offer mechanism is truncation-consistent (see Lemma B.4). Truncation-consistency plays a key role in establishing that, when choice functions are observably substitutable, the only candidate for stable and strategy-proof matching is the cumulative offer mechanism.

Observable substitutability also determines which of the two cases of Theorem 1 applies: In the first case, for a choice function that is not observably substitutable, there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.

**Theorem 1a.** If  $|H| > 1$  and the choice function of some hospital is not observably substitutable, then there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.

We prove Theorem 1a by generalizing Examples 1 and 2: Given a choice function that is not observably substitutable, we construct unit-demand choice functions for the other hospitals in such a way that either no stable outcome exists (as in Example 1) or no stable mechanism can be strategy-proof (as in Example 2). In fact, it is enough to require truncation-consistency: If the choice function of some hospital is not observably substitutable, then there exist unit-demand choice functions for the other hospitals such that no stable and truncation-consistent mechanism exists (Theorem B.1).

In the second case, we show that if the choice function of every hospital is observably substitutable, then considering cumulative offer mechanisms is sufficient for stable and strategy-proof matching. In particular:

- Every cumulative offer mechanism  $\mathcal{C}^+$  is outcome-equivalent.
- Any stable and strategy-proof mechanism is identical to a cumulative offer mechanism.

**Theorem 1b.** Suppose the choice function of each hospital is observably substitutable. Then all cumulative offer mechanisms are outcome-equivalent (i.e.,  $\mathcal{C}^+ = \mathcal{C}^{+'}$  for any two orderings  $\vdash$  and  $\vdash'$ ). Moreover, if there exists a stable and strategy-proof mechanism  $\mathcal{M}$ , then  $\mathcal{M}$  is outcome-equivalent to any cumulative offer mechanism.

*Proof.* To prove Theorem 1b, we first show that any two mechanisms that are stable and truncation-consistent have to be outcome-equivalent.

**Proposition 1.** Fix an arbitrary profile of choice functions and let  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  be stable and truncation-consistent mechanisms. Then  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  are outcome-equivalent.

*Proof.* We fix an arbitrary profile of choice functions and two stable and truncation-consistent mechanisms  $\mathcal{M}$  and  $\bar{\mathcal{M}}$ . It is immediate that when every doctor finds every contract unacceptable, all stable mechanisms yield the same outcome  $\emptyset$ . Now, suppose that  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  are outcome-equivalent for every preference profile  $\hat{\succ}$  under which at most  $N$  contracts are ranked as acceptable (aggregating across all doctors). Consider an arbitrary preference profile  $\succ$  under which  $N + 1$  contracts are ranked as acceptable. Let  $d$  be an arbitrary doctor and  $y$  be  $d$ 's least preferred acceptable contract under  $\succ_d$ .<sup>27</sup> Let  $\hat{\succ}_d$  be the truncation of  $\succ_d$  under which  $y$  is unacceptable and all contracts in  $X_d \setminus \{y\}$  that are acceptable under  $\succ_d$  remain acceptable. Now:

- If  $\mathcal{M}_d(\succ) = x \succ_d y$ , then  $\mathcal{M}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = x$ , as  $\mathcal{M}$  is truncation-consistent. But by the inductive hypothesis,  $\bar{\mathcal{M}}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = \mathcal{M}(\hat{\succ}_d, \succ_{D \setminus \{d\}})$ , and so  $\bar{\mathcal{M}}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = x$ . But then  $\bar{\mathcal{M}}_d(\succ) = x$  as  $\bar{\mathcal{M}}$  is truncation-consistent.
- Switching the roles of  $\mathcal{M}$  and  $\bar{\mathcal{M}}$ , we must also have that  $\bar{\mathcal{M}}_d(\succ) = x \succ_d y$  implies  $\mathcal{M}_d(\succ) = x$ .

The previous argument shows that  $d$  either

1. obtains the same contract (or is unemployed) under  $\mathcal{M}(\succ)$  and  $\bar{\mathcal{M}}(\succ)$ ,

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<sup>27</sup>This assumes that there is at least one such contract. Otherwise, it is immediate that  $d$  must get  $\emptyset$  under both stable mechanisms  $\mathcal{M}$  and  $\bar{\mathcal{M}}$ .

2. obtains  $y$  under  $\mathcal{M}(\succ)$  but is unemployed under  $\bar{\mathcal{M}}(\succ)$ , or
3. is unemployed under  $\mathcal{M}(\succ)$  but obtains  $y$  under  $\bar{\mathcal{M}}(\succ)$ .

Now, let

$$\begin{aligned} Z &\equiv \{z \in X \setminus \mathcal{M}(\succ) : z \succ_{d(z)} \mathcal{M}_{d(z)}(\succ)\} \\ \bar{Z} &\equiv \{z \in X \setminus \bar{\mathcal{M}}(\succ) : z \succ_{d(z)} \bar{\mathcal{M}}_{d(z)}(\succ)\} \end{aligned}$$

be the sets of all contracts that doctors prefer to  $\mathcal{M}(\succ)$  and  $\bar{\mathcal{M}}(\succ)$ , respectively. We have that  $\mathcal{M}(\succ) \cup Z = \bar{\mathcal{M}}(\succ) \cup \bar{Z}$  since, as we have argued above, for each doctor  $d \in D$ , either  $\mathcal{M}_d(\succ) = \bar{\mathcal{M}}_d(\succ)$  or  $d$  obtains his least-favorite acceptable contract under either  $\mathcal{M}$  or  $\bar{\mathcal{M}}$  and is unemployed under the other mechanism. Hence, we must have  $C^H(\mathcal{M}(\succ) \cup Z) = C^H(\bar{\mathcal{M}}(\succ) \cup \bar{Z})$ . Now, note that the stability of  $\mathcal{M}(\succ)$  implies that  $\mathcal{M}(\succ) = C^H(\mathcal{M}(\succ) \cup Z)$ : If  $C^H(\mathcal{M}(\succ) \cup Z) \subsetneq \mathcal{M}(\succ)$ , then  $\mathcal{M}(\succ)$  is not individually rational; otherwise, if  $C^H(\mathcal{M}(\succ) \cup Z) = \hat{Z} \not\subseteq \mathcal{M}(\succ)$ , then  $\hat{Z} \setminus \mathcal{M}(\succ)$  is a blocking set. Similarly,  $\bar{\mathcal{M}}(\succ) = C^H(\bar{\mathcal{M}}(\succ) \cup \bar{Z})$ . Thus,

$$\bar{\mathcal{M}}(\succ) = C^H(\bar{\mathcal{M}}(\succ) \cup \bar{Z}) = C^H(\mathcal{M}(\succ) \cup Z) = \mathcal{M}(\succ). \quad \square$$

All cumulative offer mechanisms are truncation-consistent (Lemma B.4) and stable (Theorem 6) when all choice functions are observably substitutable; hence, Proposition 1 implies that only cumulative offer mechanisms can be stable and truncation-consistent under observable substitutability.<sup>28</sup>

Since strategy-proofness implies truncation-consistency, our last observation implies Theorem 1b.  $\square$

In light of Theorem 1b, for any fixed profile of observably substitutable choice functions, all cumulative offer mechanisms are equivalent; thus, under observable substitutability, we may speak of “the” cumulative offer mechanism as a mapping  $\mathcal{C} = \mathcal{C}^+$  for any ordering  $\vdash$ .<sup>29</sup> Additionally, when choice functions are observably substitutable, the cumulative offer mechanism is outcome-equivalent to the *deferred acceptance mechanism* first described by

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<sup>28</sup>Afacan (2016) showed that when all choice functions are unilaterally substitutable, a mechanism is stable and *truncation-proof* (i.e., a doctor can never become strictly better off by truncating his preferences) if and only if it is the cumulative offer mechanism. By contrast, our proof of Theorem 1b requires only that choice functions are observably substitutable, but shows that a mechanism is stable and *truncation-consistent* if and only if it is the cumulative offer mechanism. In Online Appendix C.4, we show that when choice functions are observably substitutable, there exists a truncation-proof mechanism that is not a cumulative offer mechanism.

<sup>29</sup>More generally, we speak of the cumulative offer mechanism *cop* whenever all cumulative offer mechanisms are equivalent.

Gale and Shapley (1962); we formally define deferred acceptance mechanisms in Appendix A and prove their outcome-equivalence to cumulative offer mechanisms (Proposition A.1).

Together, Theorems 1a and 1b imply Theorem 1. If the choice function of any hospital is not observably substitutable, Theorem 1a applies and so stable and strategy-proof matching may not be possible. On the other hand, if the choice function of every hospital is observably substitutable, then the cumulative offer mechanism is the only candidate for stable and strategy-proof matching.

### 3.2 Observable Size Monotonicity

While observable substitutability is sufficient to guarantee that the cumulative offer mechanism is stable and truncation-consistent, observable substitutability by itself is not sufficient to ensure that the cumulative offer mechanism is strategy-proof for doctors. Indeed, while truncation-consistency ensures that no doctor benefits from truncating his preferences, a doctor may still benefit from manipulating his preferences in more complex ways, as we demonstrate in Example 3.<sup>30</sup>

**Example 3.** Suppose that there are two hospitals, three doctors, and a contract between each doctor and each hospital. More formally, let  $H = \{h, \hat{h}\}$ ,  $D = \{d, e, f\}$ , and  $X = \{x, \hat{x}, y, \hat{y}, z, \hat{z}\}$ , where  $d = \mathbf{d}(x) = \mathbf{d}(\hat{x})$ ,  $e = \mathbf{d}(y) = \mathbf{d}(\hat{y})$ ,  $f = \mathbf{d}(z) = \mathbf{d}(\hat{z})$ ,  $h = \mathbf{h}(x) = \mathbf{h}(y) = \mathbf{h}(z)$ , and  $\hat{h} = \mathbf{h}(\hat{x}) = \mathbf{h}(\hat{y}) = \mathbf{h}(\hat{z})$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\{x\} \succ \{y, z\} \succ \{y\} \succ \{z\} \succ \emptyset$$

and the choice function  $C^{\hat{h}}$  of  $\hat{h}$  be induced by the preference relation

$$\{\hat{y}\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \emptyset.$$

Note that both choice functions are observably substitutable. Meanwhile, let the preferences of the doctors be given by

$$\begin{aligned} \succ_d &: \hat{x} \succ x \succ \emptyset \\ \succ_e &: y \succ \hat{y} \succ \emptyset \\ \succ_f &: \hat{z} \succ z \succ \emptyset. \end{aligned}$$

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<sup>30</sup>Example 3 is based on the construction used in the proof of Theorem 12 of Hatfield and Milgrom (2005), which showed the importance of size monotonicity for strategy-proof matching.

Consider the cumulative offer mechanism  $\mathcal{C}^+$  with ordering  $x \vdash \hat{x} \vdash y \vdash \hat{y} \vdash z \vdash \hat{z}$ . In the first step of  $\mathcal{C}^+$ , we have that  $U^1 = \{\hat{x}, y, \hat{z}\}$  and so  $A^1 = \{\hat{x}\}$  and  $C^H(A^1) = \{\hat{x}\}$ . In the second step,  $U^2 = \{y, \hat{z}\}$  and so  $A^2 = \{\hat{x}, y\}$  and  $C^H(A^2) = \{\hat{x}, y\}$ . In the third step,  $U^3 = \{\hat{z}\}$  and so  $A^3 = \{\hat{x}, y, \hat{z}\}$  and  $C^H(A^3) = \{y, \hat{z}\}$ . In the fourth step,  $U^4 = \{x\}$  and so  $A^4 = \{\hat{x}, y, \hat{z}, x\}$  and  $C^H(A^4) = \{\hat{z}, x\}$ . In the fifth step,  $U^5 = \{\hat{y}\}$  and so  $A^5 = \{\hat{x}, y, \hat{z}, x, \hat{y}\}$  and  $C^H(A^5) = \{x, \hat{y}\}$ . In the sixth step,  $U^6 = \{z\}$  and so  $A^6 = \{\hat{x}, y, \hat{z}, x, \hat{y}, z\}$  and  $C^H(A^6) = \{x, \hat{y}\}$ . Finally, in the seventh step,  $U^7$  is empty and so the algorithm terminates; the outcome is  $\{x, \hat{y}\}$ .

However,  $\mathcal{C}^+$  is not strategy-proof: Consider the case when  $f$ 's preferences are given by

$$\hat{\succ}_f : z \succ \emptyset;$$

that is,  $f$  “drops”  $\hat{z}$  from his preferences. For these preferences, the first two steps of the algorithm are identical to those when  $f$ 's preferences are  $\hat{z} \succ_f z \succ_f \emptyset$ . But in the third step,  $U^3 = \{z\}$ , and so  $A^3 = \{\hat{x}, y, z\}$  and  $C^H(A^3) = \{\hat{x}, y, z\}$ . Now, in the fourth step,  $U^4$  is empty and so the algorithm terminates; the outcome is  $\{\hat{x}, y, z\}$ . Thus  $\mathcal{C}^+$  is not strategy-proof:  $f$  is better off (under  $\succ_f$ ) when the cumulative offer mechanism acts as if  $f$ 's preferences were  $z \hat{\succ}_f \emptyset$ .

But in fact, in Example 3, *no* stable mechanism is strategy-proof. This fact follows from Theorem 1b, but can also be seen directly: The only stable outcome under  $\succ$  is  $\{x, \hat{y}\}$ , while both  $\{x, \hat{y}\}$  and  $\{\hat{x}, y, z\}$  are stable under  $(\succ_{\{d,e\}}, \hat{\succ}_f)$ . Thus, to be strategy-proof, a stable mechanism  $\mathcal{M}$  must have  $\mathcal{M}(\succ_{\{d,e\}}, \hat{\succ}_f) = \{x, \hat{y}\}$ . But consider  $\hat{\succ}_d: \hat{x} \succ \emptyset$ : under  $(\hat{\succ}_d, \succ_e, \hat{\succ}_f)$ , the only stable outcome is  $\{\hat{x}, y, z\}$ . But then  $\mathcal{M}$  is still not strategy-proof, as  $d$  is better off (under  $\succ_d$ ) when  $\mathcal{M}$  acts as if  $d$ 's preferences were  $\hat{\succ}_d$ .

However, when all hospitals' choice functions are substitutable and size monotonic, the cumulative offer mechanism is strategy-proof (Hatfield and Milgrom, 2005). Here, we introduce a weakening of the size monotonicity condition that plays a crucial role in our characterization result.

**Definition 3.** A choice function  $C^h$  exhibits an observable violation of size monotonicity if there exists an observable offer process  $(x^1, \dots, x^M)$  for  $h$  such that  $|C^h(\{x^1, \dots, x^M\})| < |C^h(\{x^1, \dots, x^{M-1}\})|$ . A choice function  $C^h$  is *observably size monotonic* if it does not exhibit an observable violation of size monotonicity.

Our next result shows that, for any unital profile of classes, observable size monotonicity is necessary to guarantee the existence of a stable and strategy-proof mechanism.

**Theorem 2.** *If  $|H| > 1$  and the choice function of some hospital is not observably size monotonic, then there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.*

We prove Theorem 2 by generalizing Example 3: Given a choice function that is observably substitutable but not observably size monotonic, we construct unit-demand choice functions for the other hospitals in such a way that no stable mechanism can be strategy-proof.

### 3.3 Non-Manipulability via Contractual Terms

Theorem 1 shows that observable substitutability is necessary for stable and strategy-proof matching and, moreover, the only candidate for a stable and strategy-proof mechanism is the cumulative offer mechanism. Theorem 2 then shows that observable size monotonicity is also necessary to guarantee that the cumulative offer mechanism is stable and strategy-proof. One might expect these two conditions to be sufficient for stable and strategy-proof matching;<sup>31</sup> however, as our next example shows, an additional condition is needed to complete our characterization of when a stable and strategy-proof mechanism is guaranteed to exist. We introduce our third condition, non-manipulability via contractual terms, immediately following Example 4.

**Example 4.** Consider a setting in which  $H = \{h\}$ ,  $D = \{d, e\}$ , and  $X = \{x, \hat{x}, y, \hat{y}\}$ , with  $h(x) = h(\hat{x}) = h(y) = h(\hat{y}) = h$ ,  $d(x) = d(\hat{x}) = d$  and  $d(y) = d(\hat{y}) = e$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\{\hat{y}\} \succ \{\hat{x}\} \succ \{x, y\} \succ \{x\} \succ \{y\} \succ \emptyset.$$

The choice function  $C^h$  is observably substitutable and observably size monotonic.

If the preferences of the doctors are given by

$$\begin{aligned} \succ_d &: \hat{x} \succ x \succ \emptyset \\ \succ_e &: y \succ \hat{y} \succ \emptyset, \end{aligned}$$

then the cumulative offer mechanism produces the outcome  $\{\hat{y}\}$ . However, if  $d = d(x)$  reports his preferences as  $x \succ \emptyset$ , the cumulative offer mechanism produces the outcome  $\{x, y\}$ , under which  $d$  is strictly better off. Hence, the cumulative offer mechanism is not strategy-proof. Thus, by Theorem 1b, we see that no stable and strategy-proof mechanism exists.

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<sup>31</sup>Indeed, observable substitutability and observable size monotonicity are sufficient for the cumulative offer mechanism to be stable and strategy-proof in the settings of many-to-one matching without contracts and many-to-one matching with transfers.

Example 4 shows that having choice functions that *behave* substitutably and size monotonically under the cumulative offer mechanism do *not* ensure that the cumulative offer mechanism is stable and strategy-proof, and indeed may not be sufficient to ensure the existence of a stable and strategy-proof mechanism at all. In fact, the choice function  $C^h$  in Example 4 is substitutable; hence, not even substitutability and observable size monotonicity are sufficient. Nor (as we show in Online Appendix C.3) are size monotonicity and observable substitutability sufficient.

In Example 4, doctor  $d$  can profitably manipulate his preferences by just reordering his preferences over contracts with hospital  $h$ ; our third and final condition rules out such manipulations.

**Definition 4.** The choice function  $C^h$  of hospital  $h$  is *manipulable by doctor  $d$  via contractual terms (absent other hospitals)*, if there is a strict ordering  $\vdash$ , a preference profile  $\succ$  for doctors under which only contracts with  $h$  are acceptable, and a preference relation  $\hat{\succ}_d$  for  $d$  under which only contracts with  $h$  are acceptable such that

$$C^+(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d C^+(\succ).$$

If the choice function  $C^h$  of hospital  $h$  is manipulable by some doctor  $d$  via contractual terms, we say that the choice function  $C^h$  of hospital  $h$  is *manipulable via contractual terms*; otherwise, we say that the choice function  $C^h$  of hospital  $h$  is *non-manipulable via contractual terms*.

In contrast to observable substitutability and observable size monotonicity, the condition that a choice function be non-manipulable via contractual terms does not have any counterpart in pure matching settings: In a pure matching setting with  $h$  as the only hospital, each doctor has only one contract with hospital  $h$ , and so that doctor should rank this contract as acceptable if and only if it is preferred to the outside option.

Note that by Theorem 1b, when choice functions are observably substitutable, any stable and strategy-proof mechanism has to coincide with the cumulative offer mechanism. Hence, when a choice function is observably substitutable, the non-manipulability via contractual terms of the choice function of  $h$  essentially requires that the only candidate for a stable and strategy-proof mechanism, the cumulative offer mechanism, is strategy-proof in a fictitious economy where  $h$  is the *only* available employer. As we show in Theorem 3, the necessity of such a condition is straightforward; nevertheless, and surprisingly, this condition plays a key role in completing our characterization of when a stable and strategy-proof mechanism can be guaranteed to exist.



**Theorem 3.** *If  $|H| > 1$  and the choice function of some hospital is manipulable via contractual terms, then there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.*

*Proof.* First, note that if  $C^h$  is not observably substitutable, then Theorem 1a implies that there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists; thus, we assume that  $C^h$  is observably substitutable. By assumption, there exists a preference profile  $\succ$  under which only contracts with  $h$  are acceptable to  $d$ , a doctor  $d \in D$ , and a preference relation  $\hat{\succ}_d$  under which only contracts with  $h$  are acceptable to  $d$  such that  $\mathcal{C}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{C}(\succ)$ . Since  $C^h$  is observably substitutable, Theorem 1b implies that for any stable and strategy-proof mechanism  $\mathcal{M}$ , we have  $\mathcal{M}(\succ) = \mathcal{C}(\succ)$  and  $\mathcal{M}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = \mathcal{C}(\hat{\succ}_d, \succ_{D \setminus \{d\}})$ . Hence,  $\mathcal{M}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{M}(\succ)$ , contradicting the strategy-proofness of  $\mathcal{M}$ .  $\square$

### 3.4 Characterization

In this subsection, we establish that the three necessary conditions for the existence of a stable and strategy-proof mechanism that we have introduced are jointly sufficient for the cumulative offer mechanism to be stable and strategy-proof. Combined with our necessity results, we obtain the following characterization.

**Theorem 4** (CHARACTERIZATION OF STABLE AND STRATEGY-PROOF MATCHING). *Let  $\mathcal{C}$  be a unital profile of classes and suppose that  $|H| > 1$ . The following are equivalent:*

- (i) *For all  $h \in H$ , and for all  $C^h \in \mathcal{C}^h$ , the choice function  $C^h$  is observably substitutable, observably size monotonic, and non-manipulable via contractual terms.*
- (ii) *A stable and strategy-proof mechanism is guaranteed to exist for  $\mathcal{C}$ .*
- (iii) *Any cumulative offer mechanism is stable and strategy-proof for  $\mathcal{C}$ .*

*Furthermore, if the mechanism  $\mathcal{M}$  is stable and strategy-proof for each  $C \in \mathcal{C}$ , all cumulative offer mechanisms are equivalent for each  $C \in \mathcal{C}$  and  $\mathcal{M} = \mathcal{C}$ .*

Theorem 4 implies that market designers' reliance on cumulative offer/deferred acceptance mechanisms derives from the fundamental structure of matching: Whenever the very existence of a stable and strategy-proof mechanism can be guaranteed, any such mechanism is equivalent to the cumulative offer mechanism. In particular, any cumulative offer mechanism is stable and strategy-proof whenever both properties can be guaranteed to be satisfied jointly. This

finding provides an important justification for the use of the cumulative offer mechanism when only limited information about hospitals’ preferences is available.

Before discussing the intuition behind the sufficiency part of our characterization result (i.e., that (i) of Theorem 4 implies (iii) of Theorem 4), some remarks are in order:

- First, our conditions for the existence of a stable and strategy-proof mechanism can be checked independently at each hospital, and do not depend on subtle interactions between hospitals’ choice functions. In particular, it is possible to check our non-manipulability condition efficiently by considering only a small set of possible misreports by doctors; see Section 3.6.
- Second, by virtue of having a maximal domain characterization for the existence of a stable and strategy-proof mechanism, our three conditions subsume all previously known sufficient conditions for the existence of stable and strategy-proof mechanisms. In particular, any choice function that either
  1. is unilaterally substitutable and size monotonic (Hatfield and Kojima, 2010),
  2. is induced by slot-specific priorities (Kominers and Sönmez, 2016), or
  3. has a substitutable and size monotonic completion (Hatfield and Kominers, 2019)

must be observably substitutable, observably size monotonic, and non-manipulable via contractual terms; we discuss these connections in more detail in Section 3.5. Moreover, as we discuss in Example 5 later in this section, there exist choice functions for which our sufficient conditions apply but for which the existence of a stable and strategy-proof mechanism could not heretofore be guaranteed. Hence, the combination of observable substitutability, observable size monotonicity, and non-manipulability via contractual terms is strictly weaker than any of the previously known sets of conditions guaranteeing the existence of a stable and strategy-proof mechanism.

- Third, it is straightforward to check that our characterization is minimal in the sense that the three conditions on hospitals’ choice functions are independent: Example 1 shows that the combination of observable size monotonicity and non-manipulability via contractual terms do not imply observably substitutability; Example 3 shows that the combination of observable substitutability and non-manipulability via contractual terms do not imply observable size monotonicity; Example 4 shows that the combination of observable substitutability and observable size monotonicity do not imply non-manipulability via contractual terms.

We now discuss the proof of the sufficiency part of Theorem 4. One way to read our result is that, given observable substitutability and observable size monotonicity, just ruling out manipulations that are effective in single-employer economies is sufficient to rule out effective manipulations in the full economy. Our proof starts from the assumption that hospitals' choice functions are both observably substitutable and observably size monotonic and yet, at some preference profile  $\succ$ , some doctor  $\hat{d}$  can profitably manipulate the cumulative offer mechanism by submitting  $\hat{\succ}_{\hat{d}}$  instead of  $\succ_{\hat{d}}$  because the former preference ordering yields a strictly more preferred (under  $\succ_{\hat{d}}$ ) contract  $\hat{x}$ . In the proof, we establish that the choice function of  $\hat{h} \equiv \mathbf{h}(\hat{x})$  must be manipulable via contractual terms. The idea is to remove all contracts with hospitals other than  $\hat{h}$  from  $\succ$  and  $\hat{\succ} \equiv (\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}})$ , leading to the preference profiles  $\succ'$  and  $\hat{\succ}'$ . Intuitively, deletion of contracts with other hospitals increases the competition for contracts with the one remaining hospital  $\hat{h}$ ; here, by increased competition, we mean that doctors who were matched to other hospitals under  $\hat{\succ}$  may, under  $\hat{\succ}'$ , make additional offers to  $\hat{h}$ .<sup>32</sup> Since  $\hat{d}$  preferred  $\hat{x}$  to the contract that he obtains under the cumulative offer mechanism under  $\succ$ ,  $\hat{d}$  must prefer  $\hat{x}$  to the contract, if any, that he obtains under the cumulative offer mechanism under  $\succ'$ . The more difficult part of the proof is to show that the increased competition for contracts with  $\hat{h}$  at  $\hat{\succ}'$  does not hurt  $\hat{d}$ , in the sense that  $\hat{x}$  is not rejected during the cumulative offer mechanism for  $\hat{\succ}'$ . In the proof, we consider the additional offers made under  $\hat{\succ}'$  one by one, starting with the first offer under  $\hat{\succ}'$  not made under  $\hat{\succ}$ ; we show that adding these contracts to the set of contracts available to  $\hat{h}$  under  $\hat{\succ}$  does not induce  $\hat{h}$  to reject  $\hat{x}$ .

The proof strategy we use to prove the sufficiency part of Theorem 4 differs from that used by [Hatfield and Milgrom \(2005\)](#) and [Hatfield and Kojima \(2010\)](#) to prove that the cumulative offer mechanism is strategy-proof in their settings. [Hatfield and Milgrom \(2005\)](#) showed that when hospitals' preferences are substitutable, there exists a doctor-optimal stable outcome, i.e., a stable outcome weakly preferred by every doctor to every other stable outcome; moreover, when the hospitals' preferences are in addition size monotonic, the same set of doctors is employed at every stable outcome (a result known as the *rural hospitals theorem*). These results together imply that a mechanism which always selects the doctor-optimal stable outcome, such as the cumulative offer mechanism, is strategy-proof for doctors. [Hatfield and Kojima \(2010\)](#) showed analogous results while requiring only that hospitals' preferences be unilaterally substitutable. But, as [Example 5](#) demonstrates below, even when the preferences of each hospital are observably substitutable, observably size monotonic, and non-manipulable via contractual terms, there does not necessarily exist a doctor-optimal

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<sup>32</sup>In fact, these additional offers may induce  $\hat{h}$  to reject some contracts accepted under  $\hat{\succ}$ , which may lead to some doctors employed by  $\hat{h}$  under  $\hat{\succ}$  also making additional offers to  $\hat{h}$  under  $\hat{\succ}'$ .

stable outcome.<sup>33</sup>

**Example 5.** Consider a setting in which  $H = \{h\}$ ,  $D = \{d, e, f\}$ , and  $X = \{x, y, z, \hat{x}, \hat{z}\}$ , with  $\mathbf{d}(x) = \mathbf{d}(\hat{x}) = d$ ,  $\mathbf{d}(y) = e$ ,  $\mathbf{d}(z) = \mathbf{d}(\hat{z}) = f$ , and  $\mathbf{h}(x) = \mathbf{h}(y) = \mathbf{h}(z) = \mathbf{h}(\hat{x}) = \mathbf{h}(\hat{z}) = h$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\begin{aligned} \{\hat{x}, z\} \succ \{\hat{z}, x\} \succ \{\hat{z}, y\} \succ \{\hat{x}, y\} \succ \{x, y\} \succ \{z, y\} \succ \{\hat{x}, \hat{z}\} \succ \{x, z\} \succ \\ \{y\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \emptyset; \end{aligned}$$

it is straightforward to check that  $C^h$  is observably substitutable, observably size monotonic, and non-manipulable via contractual terms.<sup>34,35</sup> Let the preferences of the doctors be given by

$$\begin{aligned} \succ_d : x \succ \hat{x} \succ \emptyset \\ \succ_e : y \succ \emptyset \\ \succ_f : z \succ \hat{z} \succ \emptyset. \end{aligned}$$

There does not exist a doctor-optimal stable outcome, as there are two stable outcomes— $\{\hat{x}, z\}$  and  $\{\hat{z}, x\}$ , with the former preferred by  $f = \mathbf{d}(z)$  and the latter preferred by  $d = \mathbf{d}(x)$ . Nevertheless, the cumulative offer mechanism produces a stable outcome,  $\{\hat{z}, x\}$ , and moreover the cumulative offer mechanism is strategy-proof.

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<sup>33</sup>We note also that in the setting of Example 5, we cannot use the techniques of Kamada and Kojima (2012, 2015, 2018), Kominers and Sönmez (2016), or Hatfield and Kominers (2019) to construct an auxiliary economy in which a doctor-optimal stable outcome exists since (as demonstrated in Example 7) the choice function of hospital  $h$  in Example 5 is not substitutably completable.

<sup>34</sup>In fact,  $C^h$  belongs to the class of *multi-division choice functions with flexible allotments* defined in our companion paper (Hatfield et al., 2017); there, we show that every such choice function is observably substitutable, observably size monotonic, and non-manipulable via contractual terms.

<sup>35</sup>In order to see directly that  $C^h$  is non-manipulable via contractual terms, note first that  $x$  can never be proposed and rejected in the cumulative offer mechanism: If  $\hat{z}$  is proposed, the cumulative offer mechanism will choose the outcome  $\{\hat{z}, x\}$ ; if  $\hat{z}$  is not proposed and  $y$  is proposed, the cumulative offer mechanism will choose the outcome  $\{x, y\}$ ; if  $\hat{z}$  and  $y$  are both not proposed, the cumulative offer mechanism will choose  $\{x, z\}$  if  $z$  is proposed, and  $\{x\}$  if  $z$  is not proposed. Given that  $x$  cannot be proposed and rejected in the cumulative offer mechanism, it is easy to see that  $d$  cannot profitably manipulate the cumulative offer mechanism with  $h$  as the only available hospital. Similar arguments show that  $\hat{z}$  cannot be proposed and rejected in the cumulative offer mechanism. Hence,  $f$  can also not profitably manipulate the cumulative offer mechanism with  $h$  as the only available hospital. It is clear that  $e$  cannot profitably manipulate the cumulative offer mechanism since there is just one contract associated with  $e$ .

### 3.5 Relationship with Other Conditions Sufficient for Stable and Strategy-Proof Matching

[Hatfield and Milgrom \(2005\)](#) showed that substitutability and size monotonicity (called the Law of Aggregate Demand by [Hatfield and Milgrom \(2005\)](#)) are sufficient for stable and strategy-proof matching.

Recall that substitutability requires that  $R^h(Y) \subseteq R^h(Z)$  when  $Y \subseteq Z$  for any sets of contracts  $Y$  and  $Z$ . By contrast, observable substitutability requires that  $R^h(Y) \subseteq R^h(Z)$  only if there exists an observable offer process  $(x^1, \dots, x^M)$  such that  $\{x^1, \dots, x^M\} = Z$  and  $\{x^1, \dots, x^{M-1}\} = Y$ . In pure matching settings, every offer process is observable, since each doctor can propose at most one contract with a given hospital; hence, in such settings, observable substitutability is equivalent to substitutability. Thus, in pure matching settings, [Theorem 1](#) implies that substitutability is necessary for stable and strategy-proof matching.<sup>36</sup>

The gap between substitutability and observable substitutability in the setting of many-to-one matching with contracts arises because substitutability requires that the choice function act substitutably everywhere, while observable substitutability only requires that it act substitutably when facing choices generated during a cumulative offer mechanism. [Theorem 1](#) tells us that cumulative offer mechanisms are the only candidates for stable and strategy-proof matching; moreover, [Theorem 1a](#) tells us that substitutable behavior at each step of the cumulative offer mechanism is necessary for the cumulative offer mechanism to be stable and strategy-proof. Hence, to enable stable and strategy-proof matching, we do not need substitutable behavior everywhere—as specified by substitutability—but rather only at sets that can be generated by cumulative offer mechanisms—exactly as specified by observable substitutability. And indeed, observable substitutability is strictly weaker than substitutability, as we demonstrate in [Example 6](#).

**Example 6.** Consider a setting in which  $H = \{h\}$ ,  $D = \{d, e\}$ , and  $X = \{x, \hat{x}, y\}$ , with  $h(x) = h(\hat{x}) = h(y) = h$ ,  $d(x) = d(\hat{x}) = d$  and  $d(y) = e$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\{x, y\} \succ \{y\} \succ \{\hat{x}\} \succ \{x\} \succ \emptyset.$$

It is straightforward to check that  $C^h$  is not substitutable, as  $x$  and  $y$  are complementary:  $x \in R^h(\{x, \hat{x}\})$  but  $x \notin R^h(\{x, \hat{x}, y\})$ . However, the choice function  $C^h$  is observably substitutable: in any observable offer process under  $C^h$ , once  $x$  is offered  $h$  always chooses  $x$ ,

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<sup>36</sup>The fact that substitutability is needed to guarantee the existence of stable outcomes in pure many-to-one matching settings was first noted by [Hatfield and Kojima \(2008\)](#).

and so  $\hat{x}$  can never follow  $x$  in an observable offer process. Meanwhile,  $x$  can never follow  $\hat{x}$  in an observable offer process unless it is preceded by  $y$ —and since  $\hat{x}$  is not chosen by  $h$  when  $y$  is available, there are no observable offer processes under which  $h$  chooses from precisely  $\{x, \hat{x}\}$ .

Likewise, size monotonicity requires that  $|C^h(Y)| \leq |C^h(Z)|$  when  $Y \subseteq Z$  for any sets of contracts  $Y$  and  $Z$ . By contrast, observable size monotonicity requires that  $|C^h(Y)| \leq |C^h(Z)|$  only if there exists an observable offer process  $(x^1, \dots, x^M)$  such that  $\{x^1, \dots, x^M\} = Z$  and  $\{x^1, \dots, x^{M-1}\} = Y$ . In pure matching settings, every offer process is observable, since each doctor can propose at most one contract with a given hospital; hence, in such settings, observable size monotonicity is equivalent to size monotonicity.<sup>37</sup>

Like with substitutability and observable substitutability, the gap between size monotonicity and observable size monotonicity arises because observable size monotonicity only requires that the choice function act size monotonically when facing choices generated during a cumulative offer mechanism. Then, just like with observable substitutability, Theorem 1 tells us that cumulative offer mechanisms are the only candidates for stable and strategy-proof matching; hence, for stable and strategy-proof matching, we do not need size monotonic behavior everywhere—as specified by size monotonicity—but rather only at sets that can be generated by cumulative offer mechanisms—exactly as specified by observable size monotonicity.

Following the work of [Hatfield and Milgrom \(2005\)](#), a number of papers have presented weaker conditions under which stable and strategy-proof matching can be guaranteed. First, [Hatfield and Kojima \(2010\)](#) introduced *unilateral substitutability*, which, when combined with size monotonicity, ensures that cumulative offer mechanisms are stable and strategy-proof. [Kominers and Sönmez \(2016\)](#) then introduced *slot-specific priorities*, which give rise to a class of choice functions that are not necessarily unilaterally substitutable but still ensure that cumulative offer mechanisms are stable and strategy-proof. More recently, [Hatfield and Kominers \(2019\)](#) introduced an approach based on *substitutable completion*, which gave the weakest known sufficient conditions for stable and strategy-proof matching prior to our work. In particular, the approach of [Hatfield and Kominers \(2019\)](#) subsumes the sufficient conditions of [Hatfield and Kojima \(2010\)](#) (see [Zhang \(2016\)](#) and [Kadam \(2017\)](#)) and [Kominers and Sönmez \(2016\)](#) (see [Hatfield and Kominers \(2019\)](#)). The conditions we give here necessarily subsume all previously-known conditions as they characterize when stable and strategy-proof matching can be guaranteed. The relationship between the various conditions for stable and strategy-proof matching is depicted in Figure 1.

Moreover, our conditions strictly subsume the prior work—they allow for choice functions

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<sup>37</sup>The fact that size monotonicity is needed for stable and strategy-proof matching in pure many-to-one matching settings follows from Theorem 12 of [Hatfield and Milgrom \(2005\)](#).

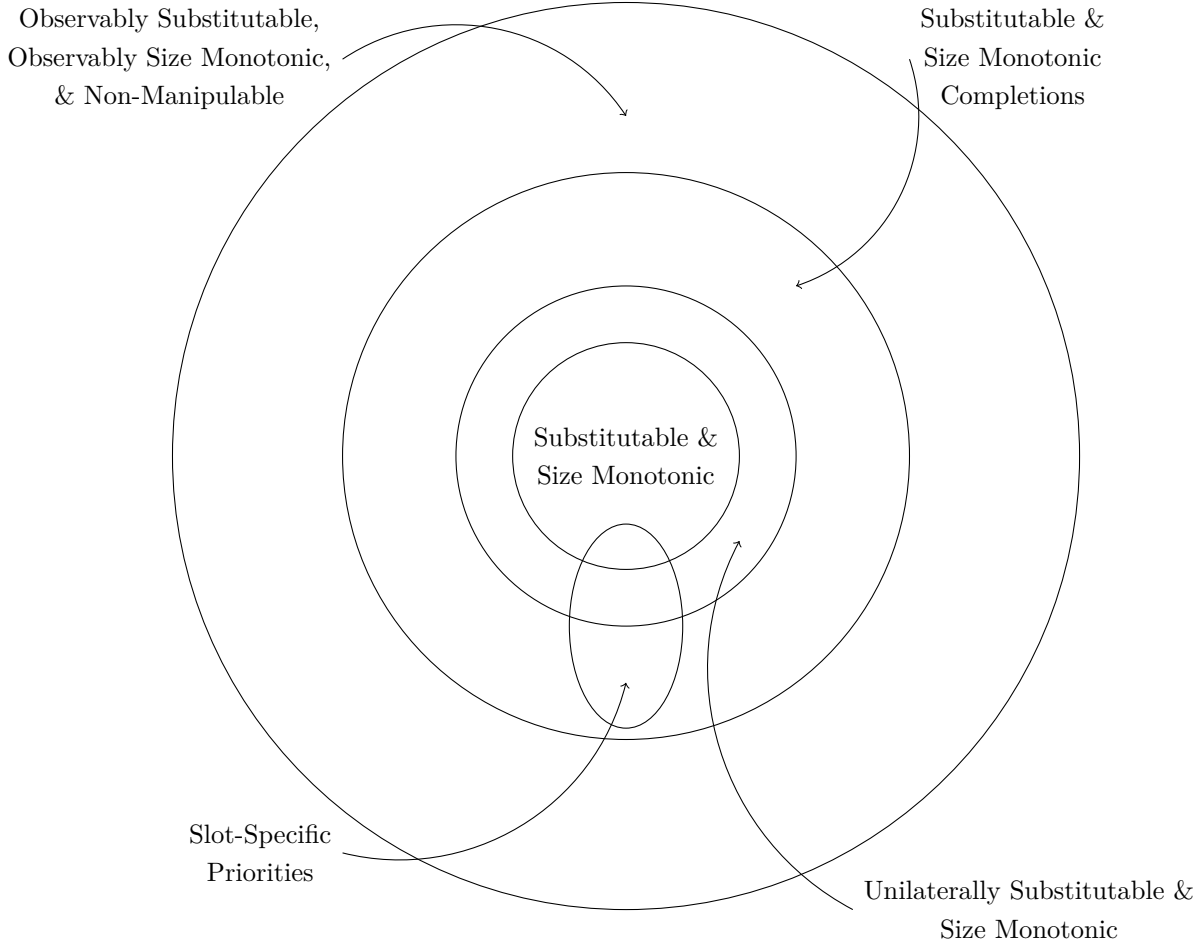


Figure 1: The relationship between sufficient conditions for the existence of a stable and strategy-proof mechanism for many-to-one matching with contracts.

under which the existence of a stable and strategy-proof mechanism could not heretofore be guaranteed.

Indeed, as we have remarked, [Hatfield and Kominers \(2019\)](#) provided the most general sufficient conditions for the guaranteed existence of stable and strategy-proof mechanisms that were known prior to our work. Specifically, [Hatfield and Kominers](#) showed that when each hospital's choice function has a substitutable and size monotonic *completion*, the cumulative offer mechanism is stable and strategy-proof. A *completion* of a choice function  $C^h$  of hospital  $h \in H$  is a choice function  $\bar{C}^h$  such that for all  $Y \subseteq X$ , either

- $\bar{C}^h(Y) = C^h(Y)$ , or
- there exist distinct  $z, \hat{z} \in \bar{C}^h(Y)$  such that  $d(z) = d(\hat{z})$ ;

that is, the completion  $\bar{C}^h$  either chooses the same set of contracts as the original choice

function or an infeasible set of contracts (i.e., a set of contracts which includes two contracts with the same doctor). In particular,  $\bar{C}^h$  can be *many-to-many*, in the sense that it may choose multiple contracts with the same doctor from a given available set. Our next example provides an example of a choice function that is observably substitutable, observably size monotonic, and non-manipulable via contractual terms—and yet does not have a substitutable completion.

**Example 7.** Consider the setting of Example 5 and let, as in Example 5, the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\begin{aligned} \{\hat{x}, z\} \succ \{\hat{z}, x\} \succ \{\hat{z}, y\} \succ \{\hat{x}, y\} \succ \{x, y\} \succ \{z, y\} \succ \{\hat{x}, \hat{z}\} \succ \{x, z\} \succ \\ \{y\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \emptyset; \end{aligned}$$

recall that  $C^h$  is observably substitutable, observably size monotonic, and non-manipulable via contractual terms.

However,  $C^h$  does not have a substitutable completion. To see this, suppose that a substitutable completion  $\bar{C}^h$  exists (with an accompanying rejection function  $\bar{R}^h$ ). By the definition of completion,  $C^h(Y) = \bar{C}^h(Y)$  for all  $Y \subseteq X$  such that  $|\mathbf{d}(Y)| = |Y|$ , i.e., for all  $Y \subseteq X$  that contain at most one contract with each doctor; hence  $R^h(Y) = \bar{R}^h(Y)$  for all such  $Y$ . Hence,

$$\begin{aligned} \hat{x} \in R^h(\{\hat{x}, y, \hat{z}\}) &\Rightarrow \hat{x} \in \bar{R}^h(\{\hat{x}, y, \hat{z}\}) \\ z \in R^h(\{x, y, z\}) &\Rightarrow z \in \bar{R}^h(\{x, y, z\}) \\ y \in R^h(\{\hat{x}, y, z\}) &\Rightarrow y \in \bar{R}^h(\{\hat{x}, y, z\}), \end{aligned}$$

as each set of contracts considered contains at most one contract with each doctor. Combining these three facts about  $\bar{R}^h$ , we have that  $\bar{C}^h(X) \subseteq \{\hat{z}, x\}$  as  $\bar{C}^h$  is substitutable. But then  $\bar{C}^h(X) = C^h(X)$ , as  $\bar{C}^h$  is a completion of  $C^h$  (as  $|\mathbf{d}(\bar{C}^h(X))|$  must equal  $|\bar{C}^h(X)|$ ); but  $C^h(X) = \{\hat{x}, z\} \not\subseteq \{\hat{z}, x\}$ , a contradiction.

Just like substitutability, weakened substitutability conditions (such as unilateral substitutability and substitutable completability) impose restrictions on the behavior of choice functions on sets that cannot be reached by any cumulative offer mechanism. For instance, in Example 7, substitutable completability requires that  $\bar{C}^h$  act substitutably on  $\{\hat{x}, y, \hat{z}\}$ ,  $\{x, y, z\}$ ,  $\{\hat{x}, y, z\}$ , and  $X$  even though the set  $X$  will never be available to  $h$  during any cumulative offer mechanism. Our characterization result is sharp exactly because observable substitutability requires (as do our other conditions) that choice functions act substitutably during cumulative offer mechanisms without any further restrictions.



### 3.6 On Non-Manipulability via Contractual Terms

Checking observable substitutability and observable size monotonicity is roughly analogous to checking the classical substitutability and size monotonicity conditions; by contrast, there is no classical counterpart to our non-manipulability via contractual terms condition. Nevertheless, as we now explain, it is not difficult to check the non-manipulability via contractual terms condition in practice.

In fact, under observable substitutability, verifying non-manipulability via contractual terms requires only checking a very specific and small class of manipulations: When the choice function of a hospital is observably substitutable, it is sufficient to check whether a doctor  $d$  can gain from a “small” misrepresentation of his true preferences—either by adding a contract to the beginning of his preference list or by removing a contract from the beginning of his preference list.

**Proposition 2.** *If  $C^h$  is a choice function for hospital  $h$  that is observably substitutable and manipulable by doctor  $d$  via contractual terms, then there exists a preference profile  $\succ$  and preferences  $\hat{\succ}_d$  under which only contracts with  $h$  are acceptable, with  $\succ_d$  of the form*

$$\succ_d : z^1 \succ \dots \succ z^M,$$

and  $\hat{\succ}_d$  of the form

$$\hat{\succ}_d : z^0 \succ z^1 \succ \dots \succ z^M,$$

such that either

1.  $\mathcal{C}_d(\succ_d, \succ_{D \setminus \{d\}}) = \emptyset$  while  $\mathcal{C}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \emptyset$ , or
2.  $\mathcal{C}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = \emptyset$  while  $\mathcal{C}_d(\succ_d, \succ_{D \setminus \{d\}}) \hat{\succ}_d \emptyset$ .

Thus, once one has verified that some choice function  $C^h$  is observably substitutable, Proposition 2 is a useful sufficient condition for non-manipulability via contractual terms because it provides us with a much smaller class of possible manipulations to consider. Indeed, in our companion paper (Hatfield et al., 2017), we use Proposition 2 to establish that the cumulative offer mechanism is stable and strategy-proof for *multi-division choice functions with flexible allotments*—even though such choice functions do not in general satisfy any pre-existing sets of sufficient conditions for stable and strategy-proof matching.

Finally, we note that—even though it may not be immediately evident from Definition 4—manipulability via contractual terms is a property that, like (observable) substitutability and size monotonicity, relies only on the choice function of a hospital  $h$ . In particular, in Online

Appendix E, we show that we can reformulate manipulability via contractual terms of  $C^h$  as a form of “inconsistency” in how  $C^h$  chooses *across* certain pairs of observable offer processes.

### 3.7 Restrictions on Doctors’ Preferences

Our main results assume, in principle, a large amount of richness in the space of doctor preferences; that is, our main results assume that each doctor can have any preference ordering over contracts involving that doctor. In many real-world settings, however, there is structure to the set of contracts that limits the preference orderings a doctor could have. For instance, suppose a contract  $x$  (where  $\mathbf{d}(x) = d$  and  $\mathbf{h}(x) = h$ ) specifies identical contractual terms to a contract  $\hat{x}$  except for a higher wage; in this case, it is natural to presume that  $x \succ_d \hat{x}$ . Thus, in this section, we generalize our results to the case in which *a priori* knowledge exists regarding a doctor  $d$ ’s preferences over contracts with a particular hospital  $h$ . We show that in order to guarantee the existence of a stable and strategy-proof mechanism, our three key properties need hold only for offer processes consistent with doctor preferences not ruled out by our *a priori* knowledge.

Let the class of all (strict) preferences for doctor  $d$  over  $X_d$  and the outside option  $\emptyset$  be denoted by  $\mathcal{P}_d$ , and let  $\mathcal{P} \equiv \times_{d \in D} \mathcal{P}_d$ . We denote the restriction of  $\succ_d$  to contracts only involving hospital  $h$  and the outside option (i.e., to  $(X_h \cap X_d) \cup \{\emptyset\}$ ) by  $\succ_d^h$ ; let the class of all such preferences be denoted by  $\mathcal{P}_d^h$ . We say that  $\mathcal{Q}_d^h \subseteq \mathcal{P}_d^h$  is a *subclass of preferences of doctor  $d$  over contracts with  $h$*  if

- there exists a contract  $x \in X_d \cap X_h$  and a preference ordering  $\succ_d^h \in \mathcal{Q}_d^h$  such that  $x \succ_d^h \emptyset$ , i.e., there exists some contract between  $d$  and  $h$  that is acceptable to  $d$  under some preference ordering in  $\mathcal{Q}_d^h$ , and
- if  $x^1 \succ_d \dots \succ_d x^M \succ_d \emptyset$  is in  $\mathcal{Q}_d^h$ , then, for all  $m = 0, \dots, M$ , we have that  $x^1 \succ_d \dots \succ_d x^m \succ_d \emptyset$  is in  $\mathcal{Q}_d^h$ , i.e., for any preference ordering in  $\mathcal{Q}_d^h$ , there exists another preference ordering in  $\mathcal{Q}_d^h$  with an identical ordering over contracts and in which fewer contracts with  $h$  are acceptable.<sup>38</sup>

To denote that  $\mathcal{Q}_d^h$  is a subclass of  $\mathcal{P}_d^h$ , we write  $\mathcal{Q}_d^h \sqsubseteq \mathcal{P}_d^h$ . Moreover,  $\mathcal{Q}_d$  is the *subclass of preferences for doctor  $d$  generated by  $\{\mathcal{Q}_d^h\}_{h \in H}$*  (where  $\mathcal{Q}_d^h \sqsubseteq \mathcal{P}_d^h$  for each  $h \in H$ ) if

$$\mathcal{Q}_d = \{\succ_d \in \mathcal{P}_d : \text{for each } h \in H, \succ_d^h \in \mathcal{Q}_d^h\};$$

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<sup>38</sup>The  $m = 0$  case ensures that there exist preference relations for  $d$  under which no contract with hospital  $h$  is acceptable.

we say that  $\mathcal{Q}_d$  is a *subclass of preferences for doctor  $d$*  if there exists some  $\{\mathcal{Q}_d^h\}_{h \in H}$  such that  $\mathcal{Q}_d$  is the subclass of preferences for doctor  $d$  generated by  $\{\mathcal{Q}_d^h\}_{h \in H}$  (where  $\mathcal{Q}_d^h \subseteq \mathcal{P}_d^h$  for each  $h \in H$ ). Finally, we say that  $\mathcal{Q}$  is a *subclass of preferences* if there exists a subclass of preferences  $\mathcal{Q}_d$  for each  $d \in D$  such that  $\mathcal{Q} = \times_{d \in D} \mathcal{Q}_d$ ; we denote that  $\mathcal{Q}$  is a subclass of preferences by  $\mathcal{Q} \subseteq \mathcal{P}$ .

Intuitively, a subclass of preferences  $\mathcal{Q}$  is a subset of  $\mathcal{P}$  such that:

- Restrictions on doctors' preferences are independent across doctors, i.e., fixing one doctor's preferences does not impose any additional restrictions on other doctors' preferences.
- Restrictions on doctors' preferences are independent across hospitals for each doctor, i.e., fixing one doctor's preferences over contracts at a given hospital does not impose any additional restrictions on that doctor's preferences at other hospitals.
- No assumptions are made about which contracts are acceptable, that is, if  $x^1 \succ_d \dots \succ_d x^M \succ_d \emptyset$  is in  $\mathcal{Q}_d$ , then, for all  $m = 0, \dots, M$ , we have that  $x^1 \succ_d \dots \succ_d x^m \succ_d \emptyset$  is in  $\mathcal{Q}_d$ .<sup>39</sup>

An offer process  $\mathbf{x} = (x^1, \dots, x^M)$  for  $h$  can be *generated from*  $\mathcal{Q} \subseteq \mathcal{P}$  if there exists  $\succ \in \mathcal{Q}$  such that, for all  $i, j \leq M$  such that  $i < j$  and  $\mathbf{d}(x^i) = \mathbf{d}(x^j)$ , we have that  $x^i \succ_d x^j$  and  $x^j \succ_d \emptyset$ .

We can now state generalizations of our three main definitions to allow for the possibility that only subclasses of preferences need to be considered.

**Definition 5.** A choice function  $C^h$  *exhibits an observable violation of substitutability on*  $\mathcal{Q}$  if there exists an observable offer process  $(x^1, \dots, x^M)$  for  $h$  generated from  $\mathcal{Q}$  such that  $R^h(\{x^1, \dots, x^{M-1}\}) \setminus R^h(\{x^1, \dots, x^M\}) \neq \emptyset$ . A choice function  $C^h$  is *observably substitutable on*  $\mathcal{Q}$  if it does not exhibit an observable violation of substitutability on  $\mathcal{Q}$ .

**Definition 6.** A choice function  $C^h$  *exhibits an observable violation of size monotonicity on*  $\mathcal{Q}$  if there exists an observable offer process  $(x^1, \dots, x^M)$  for  $h$  generated from  $\mathcal{Q}$  such that  $|C^h(\{x^1, \dots, x^M\})| < |C^h(\{x^1, \dots, x^{M-1}\})|$ . A choice function  $C^h$  is *observably size monotonic on*  $\mathcal{Q}$  if it does not exhibit an observable violation of size monotonicity on  $\mathcal{Q}$ .

**Definition 7.** The choice function  $C^h$  of hospital  $h$  is *manipulable by doctor  $d$  via contractual terms (absent other hospitals) on*  $\mathcal{Q}$ , if there is a strict ordering  $\vdash$ , a preference profile  $\succ \in \mathcal{Q}$

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<sup>39</sup>In particular, it is possible that  $d$  does not find any contract acceptable.

for doctors under which only contracts with  $h$  are acceptable, and a preference relation  $\hat{\succ}_d \in \mathcal{Q}_d$  under which only contracts with  $h$  are acceptable such that

$$\mathcal{C}^+(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{C}^+(\succ).$$

If the choice function  $C^h$  of hospital  $h$  is manipulable by some doctor  $d$  via contractual terms (absent other hospitals) on  $\mathcal{Q}$ , we say that the choice function  $C^h$  of hospital  $h$  is *manipulable via contractual terms (absent other hospitals) on  $\mathcal{Q}$* .

Note that, by taking  $\mathcal{Q} = \mathcal{P}$ , we recover our original definitions; for instance, if  $C^h$  is observably substitutable on  $\mathcal{P}$ , then  $C^h$  is observably substitutable. However, we can construct examples, such as Example 8 in the sequel, for which  $\mathcal{Q}$  is a strict subset of  $\mathcal{P}$  and there exist choice functions that are not observably substitutable but are observably substitutable on  $\mathcal{Q}$ .

**Example 8.** Consider a setting in which  $H = \{h\}$ ,  $D = \{d, e\}$ , and  $X = \{x, \hat{x}, y\}$ , with  $h(x) = h(\hat{x}) = h(y) = h$ ,  $d(x) = d(\hat{x}) = d$ , and  $d(y) = e$ . Let the choice function  $C^h$  of  $h$  be induced by the preference relation

$$\{\hat{x}, y\} \succ \{\hat{x}\} \succ \{x\} \succ \emptyset.$$

The choice function  $C^h$  is not observably substitutable as, for the observable offer process  $(y, \hat{x})$ , we have that  $y \notin C^h(\{y\})$  but  $y \in C^h(\{\hat{x}, y\})$ . However,  $C^h$  is observably substitutable on  $\mathcal{Q} = \{x \succ \hat{x} \succ \emptyset, x \succ \emptyset \succ \hat{x}, \emptyset \succ x \succ \hat{x}\} \times \mathcal{P}_e$ ; <sup>40</sup> note that  $\mathcal{Q}$  is a subclass of  $\mathcal{P}$ .

For a subclass  $\mathcal{Q}$  of  $\mathcal{P}$ , a mechanism  $\mathcal{M}$  is *stable on  $\mathcal{Q}$*  if  $\mathcal{M}(\succ)$  is a stable outcome for every preference profile  $\succ \in \mathcal{Q}$ . A mechanism  $\mathcal{M}$  is *strategy-proof on  $\mathcal{Q}$*  if for every preference profile  $\succ \in \mathcal{Q}$ , and for each doctor  $d \in D$ , there does not exist a  $\hat{\succ}_d \in \mathcal{Q}_d$  such that  $\mathcal{M}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{M}(\succ)$ . We now state a generalization of Theorem 4 that allows for known restrictions on the preferences of the doctors.

**Theorem 5.** *Let  $\mathcal{C}$  be a unital profile of classes and  $\mathcal{Q}$  a subclass of  $\mathcal{P}$ , and suppose that  $|H| > 1$ . The following are equivalent:*

- *For all  $h \in H$ , and for all  $C^h \in \mathcal{C}^h$ , the choice function  $C^h$  is observably substitutable on  $\mathcal{Q}$ , observably size monotonic on  $\mathcal{Q}$ , and non-manipulable via contractual terms on  $\mathcal{Q}$ .*

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<sup>40</sup>One possible interpretation of  $\mathcal{Q}$  in this setting is that  $e$  is a physician assistant whom the hospital  $h$  would like to hire if they can hire the physician  $d$ , but it can only do so if  $d$  takes the lower-salaried contract  $\hat{x}$  instead of the higher-salaried contract  $x$  due to  $h$ 's budget constraint. Since  $x$  is the higher-salaried contract, it is natural to require that  $d$  prefers  $x$  to  $\hat{x}$ .

- A stable and strategy-proof mechanism on  $\mathcal{Q}$  is guaranteed to exist for  $\mathcal{C}$ .
- Any cumulative offer mechanism is stable and strategy-proof on  $\mathcal{Q}$  for  $\mathcal{C}$ .

Furthermore, if the mechanism  $\mathcal{M}$  is stable and strategy-proof on  $\mathcal{Q}$  for each  $C \in \mathcal{C}$ , all cumulative offer mechanisms are equivalent for each  $C \in \mathcal{C}$  and  $\mathcal{M} = \mathcal{C}$  on  $\mathcal{Q}$ .

Theorem 5 implies that *a priori* knowledge about doctors' preferences expands the set of hospital choice functions for which we can find stable and strategy-proof mechanisms—and again, any such mechanism is outcome-equivalent to the cumulative offer mechanism.

Our results here generalize the main result of [Abizada and Dur \(2017\)](#), who studied a matching model in which contracts encode a student, a college, and a scholarship amount; in their model, [Abizada and Dur \(2017\)](#) found a class of choice functions for colleges that are not observably substitutable (or even weakly substitutable in the sense of [Hatfield and Kojima \(2008\)](#)) yet for which a stable and strategy-proof mechanism is guaranteed to exist. The [Abizada and Dur \(2017\)](#) result relies on the assumption that students prefer larger scholarships—and in fact the class of college choice functions they identify is observably substitutable (and satisfies our other conditions) for any offer process generated by the subclass of students' preferences under which larger scholarships are preferred.<sup>41</sup>

## 4 Stable Outcomes and Cumulative Offer Mechanisms

The results of Section 3 show that when one is interested in the existence of a stable and strategy-proof mechanism for a unital profile of classes, attention can be restricted to the cumulative offer mechanism. This naturally leads to the question of whether the restriction to cumulative offer mechanisms is also without loss of generality when the only constraint is that a stable outcome is to be reached. To answer this question, we first introduce a weakening of the observable substitutability condition.

**Definition 8.** A choice function  $C^h$  is *observably substitutable across doctors*, if, for any observable offer process  $(x^1, \dots, x^M)$  for  $h$ , we have that if  $x \in R^h(\{x^1, \dots, x^{M-1}\}) \setminus R^h(\{x^1, \dots, x^M\})$  then  $\mathbf{d}(x) \in \mathbf{d}(C^h(\{x^1, \dots, x^{M-1}\}))$ .

Note that observable substitutability across doctors is weaker than observable substitutability given that the latter requires  $R^h(\{x^1, \dots, x^{M-1}\}) \setminus R^h(\{x^1, \dots, x^M\}) = \emptyset$  for any observable offer process  $(x^1, \dots, x^M)$ . By contrast, observable substitutability across doctors requires that whenever a hospital chooses a previously-rejected contract  $x^m$  when  $x^M$  becomes

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<sup>41</sup>See [Schlegel \(2019\)](#) for a formal derivation of this result.

available during the observable offer process  $(x^1, \dots, x^M)$ , the hospital was already choosing some contract  $x'$  with the same doctor, i.e.,  $d(x^m) = d(x')$  for some  $x' \in C^h(\{x^1, \dots, x^{M-1}\})$ .<sup>42</sup>

The first result of this section is that observable substitutability across doctors is sufficient for all cumulative offer mechanisms to be outcome-equivalent.

**Proposition 3.** *Suppose the choice function of each hospital is observably substitutable across doctors. Then all cumulative offer mechanisms are outcome-equivalent (i.e.,  $C^\vdash = C^{\vdash'}$  for any two orderings  $\vdash$  and  $\vdash'$ ).<sup>43</sup>*

Our second result shows that observable substitutability across doctors implies that the cumulative offer mechanism always produces a stable outcome.<sup>44</sup>

**Theorem 6.** *Suppose the choice function of each hospital is observably substitutable across doctors. Then the cumulative offer mechanism is stable.*

Cumulative offer and deferred acceptance mechanisms are equivalent when hospitals' choice functions are observably substitutable (Proposition A.1); however, this equivalence no longer holds when we only require that hospitals' choice functions are observably substitutable across doctors. When hospitals' choice functions are observably substitutable across doctors, a contract rejected at some step of a cumulative offer mechanism may be chosen at some later step; by contrast, a contract rejected at some step of a deferred acceptance process may never be chosen at some later step, and so the outcomes of deferred acceptance mechanisms and cumulative offer mechanisms can differ. In fact, deferred acceptance mechanisms do not necessarily produce stable outcomes when hospitals' choice functions are observably substitutable across doctors, as we demonstrate in Appendix A.

Before discussing the necessity of observable substitutability across doctors for the cumulative offer mechanism to produce stable outcomes, we discuss the relationship of observable substitutability across doctors with the Hatfield and Kojima (2010) bilateral substitutability condition, one of the weakest previously-known conditions on hospital choice functions sufficient to ensure the existence of stable outcomes. A choice function  $C^h$  is *bilaterally*

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<sup>42</sup>Hatfield and Kojima (2010) refer to this as “renegotiation,” as the hospital and doctor “renegotiate” the terms of the doctor’s employment to their mutual benefit. Such renegotiation does not take place during a cumulative offer mechanism if the choice function of a hospital is observably substitutable.

<sup>43</sup>Prior to our work, Hirata and Kasuya (2014) showed that cumulative offer mechanisms are order-independent when each firm’s choice function satisfies the Hatfield and Kojima (2010) bilateral substitutability condition, and Hatfield and Kominers (2019) showed a similar result when each firm’s choice function is substitutably completable; Proposition 3 generalizes these results to settings where each firm’s choice function is observably substitutable across doctors. In Online Appendix C.2, we show that this generalization is strict, providing an example of an observably substitutable (across doctors) choice function that is neither bilaterally substitutable nor substitutably completable.

<sup>44</sup>Building on our result here, Zhang (2016) weakens our assumption of the irrelevance of rejected contracts condition to require only an assumption of an observable irrelevance of rejected contracts condition.

*substitutable* if for every set of contracts  $Y \subseteq X$  and every pair of contracts  $x, z \in X \setminus Y$  such that  $d(x), d(z) \notin d(Y)$ , we have that if  $z \notin C^h(Y \cup \{z\})$  then  $z \notin C^h(Y \cup \{x, z\})$ . [Hatfield and Kojima \(2010\)](#) showed that bilateral substitutability of hospitals' choice functions is sufficient to ensure that for any preference profile of the doctors and any ordering of contracts, the corresponding cumulative offer mechanism yields a stable outcome (Theorem 1 of [Hatfield and Kojima, 2010](#)). It is straightforward to show that bilateral substitutability implies observable substitutability across doctors and that observable substitutability across doctors is strictly weaker than bilateral substitutability.<sup>45,46</sup> Hence, Theorem 6 implies Theorem 1 of [Hatfield and Kojima \(2010\)](#).<sup>47</sup>

The final result of this section shows that, for any unital profile of classes, observable substitutability across doctors is necessary to guarantee the stability of cumulative offer mechanisms.

**Theorem 7.** *If  $|H| > 1$  and the choice function of some hospital is not observably substitutable across doctors, then there exist unit-demand choice functions for the other hospitals such that no cumulative offer mechanism is stable.*

We might hope that observable substitutability across doctors would be necessary for the existence of stable outcomes, in the sense that if the choice function of some hospital is not observably substitutable across doctors, then there exist unit-demand choice functions for the other hospitals and preferences for the doctors such that no stable outcome exists; unfortunately, as our next example shows, this is not the case.

**Example 9.** Consider the setting in which  $H = \{h\}$ ,  $D = \{d, e, f, g\}$ , and  $X = \{w, x, \hat{x}, y, z, \hat{z}\}$ , with  $d(x) = d(\hat{x}) = d$ ,  $d(y) = e$ ,  $d(z) = d(\hat{z}) = f$ ,  $d(w) = g$ , and  $h(w) = h(x) = h(\hat{x}) = h(y) = h(z) = h(\hat{z}) = h$ . Consider the choice function  $C^h$  induced by the following preference

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<sup>45</sup>Suppose that  $C^h$  is not observably substitutable across doctors. Let  $(x^1, \dots, x^M)$  be an observable offer process for  $h$  and  $x \in \{x^1, \dots, x^M\}$  be a contract such that  $x \in R^h(\{x^1, \dots, x^{M-1}\}) \setminus R^h(\{x^1, \dots, x^M\})$  even though  $d(x) \notin d(C^h(\{x^1, \dots, x^{M-1}\}))$ . Set  $Y \equiv C^h(\{x^1, \dots, x^{M-1}\}) \cup (C^h(\{x^1, \dots, x^M\}) \setminus \{x, x^M\})$  and note that irrelevance of rejected contracts implies  $C^h(Y \cup \{x\}) = C^h(\{x^1, \dots, x^{M-1}\})$  and  $C^h(Y \cup \{x, x^M\}) = C^h(\{x^1, \dots, x^M\})$ . Since  $(x^1, \dots, x^M)$  is observable,  $d(x^M) \notin d(C^h(\{x^1, \dots, x^{M-1}\}))$ . By the construction of  $Y$ , this implies  $d(x), d(x^M) \notin d(Y)$ ; thus, we see that  $C^h$  is not bilaterally substitutable.

<sup>46</sup>As we discussed in Footnote 43, Online Appendix C.2 presents an example of an observably substitutable choice function that is not bilaterally substitutable.

<sup>47</sup>[Flanagan \(2014\)](#) introduced a condition called *cumulative offer revealed bilateral substitutability* and argues, somewhat informally, that this condition is sufficient for the cumulative offer mechanism to produce a stable outcome; we discuss the relationship of cumulative offer revealed bilateral substitutability with observable substitutability across doctors in Online Appendix D.2.

relation

$$\begin{aligned} \{w, x, z\} \succ \{w, \hat{z}\} \succ \{w, \hat{x}\} \succ \{w, x\} \succ \{w, z\} \succ \{w\} \succ \\ \{y, \hat{z}\} \succ \{y, x, z\} \succ \{y, \hat{x}\} \succ \{y, x\} \succ \{y, z\} \succ \{y\} \succ \\ \{x, z\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \emptyset. \end{aligned}$$

Consider the offer process  $(z, \hat{x}, \hat{z}, x, y, w)$ —this offer process is observable, yet  $\mathbf{d}(x) \notin C^h(\{z, \hat{x}, \hat{z}, x, y\})$  while  $\mathbf{d}(x) \in C^h(\{z, \hat{x}, \hat{z}, x, y, w\})$ . Hence, the choice function  $C^h$  is not observably substitutable across doctors.<sup>48</sup>

However, when other hospitals have observably substitutable choice functions, a stable outcome always exists. To see this, let  $\hat{C}^h$  be the choice function induced by the preference relation

$$\begin{aligned} \{w, x, z\} \succ \{w, \hat{z}\} \succ \{w, \hat{x}\} \succ \{w, x\} \succ \{w, z\} \succ \{w\} \succ \\ \{\mathbf{y}, x, z\} \succ \{\mathbf{y}, \hat{z}\} \succ \{\mathbf{y}, \hat{x}\} \succ \{\mathbf{y}, x\} \succ \{\mathbf{y}, z\} \succ \{\mathbf{y}\} \succ \\ \{x, z\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \emptyset; \end{aligned}$$

that is, consider the choice function  $\hat{C}^h$  induced by switching the ordering of  $\{y, x, z\}$  and  $\{y, \hat{z}\}$  in the preference relation that induces  $C^h$ . Note that  $\hat{C}^h$  is observably substitutable across doctors.

We claim that, if  $C^{\bar{h}}$  is observably substitutable across doctors for all  $\bar{h} \in H \setminus \{h\}$ , then the cumulative offer mechanism  $\mathcal{C}(\cdot; (\hat{C}^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}}))$  is stable with respect to  $(C^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}})$ . To see this, consider any preference profile  $\succ$  for the doctors and let  $Y = \mathcal{C}(\succ; (\hat{C}^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}}))$ ; we will show that  $Y$  is stable with respect to  $\succ$  and  $(C^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}})$ . If  $Y_h \neq \{y, x, z\}$ , then the stability of  $Y$  with respect to  $\succ$  and  $(C^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}})$  follows immediately from the stability of  $Y$  with respect to  $\succ$  and  $(\hat{C}^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}})$ . If  $Y_h = \{y, x, z\}$ , but  $Y$  is not stable with respect to  $\succ$  and  $(C^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}})$ , then there exists a blocking set  $Z$  such that  $Z_h = \{\hat{z}\}$ . However, if  $\hat{z} \in Z$  and  $z \in Y$ , then we must have that  $\hat{z} \succ_{\mathbf{d}(z)} z$  as  $Z$  is a blocking set. But we can compute directly that if  $\hat{z}$  is proposed at some step of  $\mathcal{C}(\cdot; (\hat{C}^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}}))$  before  $z$  is proposed, then it is never rejected. Thus, if  $\hat{z} \succ_{\mathbf{d}(z)} z$  we cannot have that  $z \in Y = \mathcal{C}(\succ; (\hat{C}^h, (C^{\bar{h}})_{\bar{h} \in H \setminus \{h\}}))$ .

Example 9 shows that it is not sufficient to restrict attention to cumulative offer mechanisms if we are only interested in obtaining stable outcomes.

<sup>48</sup>It is also easy to see directly that  $C^h$  is not bilaterally substitutable:  $x \notin C^h(\{z, \hat{z}, x, y\})$  but  $x \in C^h(\{z, \hat{z}, x, y, w\})$ .



## 5 Conclusion

In many real world settings, firms’ preferences are not substitutable and yet the cumulative offer mechanism (or an equivalent mechanism) is stable and strategy-proof—as demonstrated by [Kamada and Kojima \(2012, 2015\)](#) in the context of matching with regional caps, [Sönmez and Switzer \(2013\)](#) and [Sönmez \(2013\)](#) in the context of matching military cadets to branches, [Dimakopoulos and Heller \(2019\)](#) in the context of matching lawyers to entry-level positions in Germany, [Hassidim et al. \(2017\)](#) in the context of matching students to psychology graduate programs in Israel, [Aygün and Turhan \(2017\)](#) in the context of matching students to colleges in India, and [Hafalir et al. \(2019\)](#) in the context of interdistrict school choice. Furthermore, in a companion piece ([Hatfield et al., 2017](#)), we introduced a new class of choice functions, *multi-division choice functions with flexible allotments*, that allow for hospitals to have multiple divisions, where the allotment to each division depends on the set of contracts available; we showed that all such choice functions satisfy the three key conditions introduced here, but do not in general satisfy any of the previous sufficient conditions for the existence of stable and strategy-proof mechanisms.<sup>49</sup>

Our work shows that the ubiquity of cumulative offer mechanisms is not by chance: We show that when each hospital’s choice function is observably substitutable, observably size monotonic, and non-manipulable via contractual terms, the cumulative offer mechanism is the *unique* stable and strategy-proof mechanism. By contrast, if any of our three conditions fails, then there exist unit-demand choice functions for the other hospitals such that *no* stable and strategy-proof mechanism exists. Thus, our results imply that the doctor-proposing cumulative offer mechanism is an essential tool in the market designer’s toolbox because it is uniquely well-suited for many-to-one matching with contracts: whenever a stable and strategy-proof mechanism is guaranteed to exist, the cumulative offer mechanism is the unique such mechanism.

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<sup>49</sup>We further showed that the matching with regional caps model of [Kamada and Kojima \(2012, 2015\)](#) can be expressed in terms of multi-division choice functions with flexible allotments.

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## A Deferred Acceptance Mechanisms

A deferred acceptance mechanism is, like a cumulative offer mechanism, defined with respect to a strict ordering  $\vdash$  of the elements of  $X$ . For any preference profile  $\succ$ , the outcome of the deferred acceptance mechanism, denoted by  $\mathcal{D}^+(\succ)$ , is determined by the *deferred acceptance process with respect to  $\vdash$  and  $\succ$*  as follows:

**Step 0:** Initialize the set of contracts *offered* by the doctors as  $F^0 = \emptyset$  and the set of contracts *held* by the hospitals as  $E^0 = \emptyset$ .

**Step  $t \geq 1$ :** Consider the set

$$V^t \equiv \{x \in X \setminus F^{t-1} : d(x) \notin d(E^{t-1}) \text{ and } \nexists z \in (X_{d(x)} \setminus F^{t-1}) \cup \{\emptyset\} \text{ such that } z \succ_{d(x)} x\}.$$

If  $V^t$  is empty, then the algorithm terminates and the outcome is given by  $E^{t-1}$ . Otherwise, letting  $z^t$  be the highest-ranked element of  $V^t$  according to  $\vdash$ , we set  $F^t = F^{t-1} \cup \{z^t\}$ , set  $E^t = C^H(E^{t-1} \cup \{z^t\})$  and proceed to step  $t + 1$ .

A deferred acceptance process begins with no contracts offered to the hospitals (i.e.,  $F^0 = \emptyset$ ) and no contracts held by the hospitals (i.e.,  $E^0 = \emptyset$ ). Then, at each step  $t$ , we construct  $V^t$ , the set of contracts that (1) have not yet been offered, (2) are not associated to doctors with contracts currently held by hospitals, and (3) are both acceptable and the most-preferred by their associated doctors among all contracts not yet proposed. If  $V^t$  is empty, then every doctor  $d$  either has some associated contract held by some hospital, i.e.,  $d \in d(E^{t-1})$ , or has no acceptable contracts left to offer, and so the deferred acceptance process ends. Otherwise, the contract in  $V^t$  that is highest-ranked according to  $\vdash$  is offered by its associated doctor, the hospitals hold their favorite sets of contracts from those they held previously and the new offer, and the process proceeds to the next step. Note that, as the number of contracts is finite, at some step the deferred acceptance process must end.

When choice functions are observably substitutable, any deferred acceptance mechanism is equivalent to the cumulative offer mechanism.

**Proposition A.1.** *Suppose that the choice function of every hospital is observably substitutable and let  $\vdash$  be an arbitrary ordering of  $X$ . Then the cumulative offer mechanism with respect to  $\vdash$  is outcome-equivalent to the deferred acceptance mechanism with respect to  $\vdash$ .*

*Proof.* Fix a preference profile  $\succ$ . We proceed by induction on the steps  $t$  of the cumulative offer mechanism with respect to  $\vdash$  and  $\succ$  and the deferred acceptance process with respect to  $\vdash$  and  $\succ$ . We show that for each  $t$  that the set of available contracts under the cumulative

offer mechanism is the set of offered contracts under the deferred acceptance process, i.e.,  $A^t = F^t$  and that the set of contracts the hospitals choose from the available contracts is the same as the set of held contracts, i.e.,  $C^H(A^t) = E^t$ ; this shows that  $\mathcal{C}^+ = \mathcal{D}^+$ .

It is immediate that  $A^0 = \emptyset = F^0$  and that  $C^H(A^0) = C^H(\emptyset) = \emptyset = E^0$ . Thus, by way of induction, assume that  $A^{t-1} = F^{t-1}$  and that  $C^H(A^{t-1}) = E^{t-1}$ . It follows then that  $U^t = V^t$ , as  $A^{t-1} = F^{t-1}$  and  $C^H(A^{t-1}) = E^{t-1}$ ; thus, if  $y^t$  denotes the next contract proposed in the cumulative offer process and  $z^t$  denotes the next contract proposed in the deferred acceptance process, then  $y^t = z^t$ . Since  $A^{t-1} = F^{t-1}$ , it is then immediate that  $A^t = A^{t-1} \cup \{y^t\} = F^{t-1} \cup \{y^t\} = F^t$ . Finally, since the choice function of each hospital  $h$  is observably substitutable, we have that  $R^h(A^{t-1}) \setminus R^h(A^t) = \emptyset$  for all  $h \in H$ ;<sup>50</sup> thus,  $C^h(A^t) \subseteq C^h(A^{t-1}) \cup \{y^t\}$  for all  $h \in H$ . Therefore,  $C^H(A^t) \subseteq C^H(A^{t-1}) \cup \{y^t\} = E^{t-1} \cup \{y^t\}$ , where the equality follows from the inductive hypothesis. Thus, by the irrelevance of rejected contracts condition, we have that  $C^H(A^t) = C^H(E^{t-1} \cup \{y^t\}) = E^t$ .  $\square$

Our next example shows that when choice functions are not observably substitutable, a deferred acceptance mechanism may not be stable, even if choice functions are observably substitutable across doctors.

**Example 10.** Consider a setting in which  $H = \{h\}$ ,  $D = \{d, e\}$ , and  $X = \{x, \hat{x}, y, \hat{y}\}$ , with  $d(x) = d(\hat{x}) = d$ ,  $d(y) = d(\hat{y}) = e$ , and  $h(x) = h(\hat{x}) = h(y) = h(\hat{y}) = h$ .

Let the choice function  $C^h$  of  $h$  be induced by the preferences

$$\{x, y\} \succ \{\hat{x}\} \succ \{\hat{y}\} \succ \{x\} \succ \{y\} \succ \emptyset.$$

The choice function  $C^h$  is observably substitutable across doctors but not observably substitutable.

If the preferences of the doctors are given by

$$\begin{aligned} \succ_d &: x \succ \hat{x} \succ \emptyset \\ \succ_e &: \hat{y} \succ y \succ \emptyset, \end{aligned}$$

then every cumulative offer mechanism produces the (stable) outcome  $\{x, y\}$  while any deferred acceptance mechanism produces the (unstable) outcome  $\{\hat{x}\}$ .

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<sup>50</sup>The result is immediate for all  $\hat{h} \neq h(y^t)$  as  $A_{\hat{h}}^{t-1} = A_{\hat{h}}^t$ ; for  $h = h(y^t)$ , note that the subsequence of  $(y^1, \dots, y^t)$  that includes all of and only the contracts with  $h$  is an observable offer process for  $h$  as it was generated by a cumulative offer mechanism.

## B Proofs Omitted from the Main Text

We first gather some additional definitions that we use throughout our proofs. We start by introducing more general notions of offer processes and observability. An *offer process*  $\mathbf{x} = (x^1, \dots, x^M)$  is a finite sequence of distinct contracts; note that an offer process may contain contracts with many different hospitals. We denote by  $\mathbf{c}(\mathbf{x}) \equiv \{x^1, \dots, x^M\}$  the set of contracts included in the offer process  $\mathbf{x}$ . We say that  $h$  *holds* the contract  $x$  after  $\mathbf{x}$  if  $x \in C^h(\mathbf{c}(\mathbf{x}))$ ; similarly, we say that  $h$  *holds*, or *employs*, the doctor  $d$  after  $\mathbf{x}$  if  $d \in \mathbf{d}(C^h(\mathbf{c}(\mathbf{x})))$ .

Fixing the choice functions of the hospitals, we say that an offer process  $\mathbf{x} = (x^1, \dots, x^M)$  is *observable* if  $\mathbf{d}(x^m) \notin \mathbf{d}(C^H(\{x^1, \dots, x^{m-1}\}))$  for all  $m = 1, \dots, M$ . We use the term observable as, during a cumulative offer mechanism, only doctors who do not have contracts currently held by a hospital are allowed to make offers. Hence, an observable offer process is an offer process that could be generated by a cumulative offer mechanism (in the sense that if  $\mathbf{x} = (x^1, \dots, x^M)$  is observable, then there exists an ordering  $\vdash$  and preference profile  $\succ$  such that  $x^m = y^m$ , where  $y^m$  is the  $m^{\text{th}}$  proposed contract in the cumulative offer mechanism with respect to  $\vdash$  and  $\succ$ ).

An offer process  $\mathbf{x} = (x^1, \dots, x^M)$  is *compatible with a preference profile*  $\succ$  if

1.  $\mathbf{x}$  is observable, and,
2. for all  $m \in \{1, \dots, M\}$ ,
  - $x^m \succ_{\mathbf{d}(x^m)} \emptyset$  and,
  - if  $x \succ_{\mathbf{d}(x^m)} x^m$ , then  $x \in \{x^1, \dots, x^{m-1}\}$ .

An offer process  $\mathbf{x}$  is *complete with respect to*  $\succ$  and  $C = (C^h)_{h \in H}$  if  $\mathbf{x}$  is compatible with  $\succ$  and, for all  $d \notin \mathbf{d}(C^H(\mathbf{c}(\mathbf{x})))$ , if  $y \in X_d \setminus \mathbf{c}(\mathbf{x})$ , then  $\emptyset \succ_d y$ . We use the term complete as when hospitals choose from  $\mathbf{c}(\mathbf{x})$ , each doctor is either employed or has proposed every contract he finds acceptable.

Finally, an offer process  $\mathbf{y} = (y^1, \dots, y^M)$  is *weakly observable* if, for all  $m \leq M$ ,  $\mathbf{d}(y^m) \notin \mathbf{d}(C^h(y^m)(\{y^1, \dots, y^{m-1}\}))$ ; that is, an offer process is weakly observable if no doctor  $d$  makes an offer to a hospital that currently employs  $d$ . Note that the weak observability of  $\mathbf{y}$  is equivalent to requiring that, for any  $h \in H$ , the subsequence of  $\mathbf{y}$  that contains only the contracts involving  $h$  is observable. In particular, if  $\mathbf{y} = (y^1, \dots, y^M)$  is weakly observable and  $C^h$  is observably substitutable for all  $h \in H$ , then  $R^H(\{y^1, \dots, y^{M-1}\}) \subseteq R^H(\mathbf{c}(\mathbf{y}))$ .<sup>51</sup>

<sup>51</sup>We denote by  $R^H(Y) \equiv \cup_{h \in H} R^h(Y)$  the set of contracts rejected by the full set of hospitals from a set of contracts  $Y \subseteq X$ .



Next, given a preference profile  $\succ$  and a set of contracts  $Y \subseteq X$ , we define the *restriction*  $\succ^Y$  of  $\succ$  to  $Y$  as follows:

1. For all  $x, y \in Y$  such that  $\mathbf{d}(x) = \mathbf{d}(y)$ ,  $x \succ_{\mathbf{d}(x)}^Y y$  if and only if  $x \succ_{\mathbf{d}(x)} y$ .
2. For all  $x \in X$ ,  $x \succ_{\mathbf{d}(x)}^Y \emptyset$  if and only if  $x \succ_{\mathbf{d}(x)} \emptyset$  and  $x \in Y$ .

We say that the preference profile  $\succ$  is *consistent with*  $Y$  if the following conditions hold:

1. If  $x \in Y$ , then  $x \succ_{\mathbf{d}(x)} \emptyset$ .
2. If  $x \in X_{\mathbf{d}(x)} \setminus Y$ , then  $\emptyset \succ_{\mathbf{d}(x)} x$ .

In other words, the preference profile  $\succ$  is consistent with  $Y$  if a contract  $y$  is acceptable to  $\mathbf{d}(y)$  if and only if  $y \in Y$ . We also say that the preferences  $\succ$  are *consistent with*  $(\mathbf{y}, Y)$  if  $\succ$  is consistent with  $Y$  and  $\mathbf{y}$  is compatible with  $\succ$ .

An offer process  $\mathbf{y} = (y^1, \dots, y^M)$  is *weakly compatible with a preference profile*  $\succ$ , if, for all  $m \in \{1, \dots, M\}$ ,  $h \in H$ , and  $d \in D$ ,

1.  $y^m \in X_d$  implies that  $y^m \succ_d \emptyset$ , and
2. for any contract  $y \in (X_h \cap X_d) \setminus \{y^m\}$  such that  $y \succ_d y^m$ ,  $y \in \{y^1, \dots, y^{m-1}\}$ .

That is, an offer process  $\mathbf{y}$  is weakly compatible with a preference profile  $\succ$  if, for each  $y^m \in \mathbf{c}(\mathbf{y})$ ,

1.  $y^m$  is an acceptable contract, and
2. the doctor making the offer  $y^m$  prefers  $y^m$  to every other contract *with the same hospital* that has not yet been offered.

We can *combine* two offer processes  $\mathbf{y} = (y^1, \dots, y^M)$  and  $\mathbf{z} = (z^1, \dots, z^N)$  as  $(\mathbf{y}, \mathbf{z}) = (w^1, \dots, w^K)$  where

- $K = |\mathbf{c}(\mathbf{y}) \cup \mathbf{c}(\mathbf{z})|$ ,
- $w^k = y^k$  for all  $k \leq M$ , and
- $w^k = z^{\ell_k}$  for  $k > M$ , where  $\ell_k \equiv \min\{\ell \in \{1, \dots, N\} : z^\ell \notin \{w^1, \dots, w^{k-1}\}\}$ .

Our first lemma establishes a condition under which we can combine two different weakly observable offer processes to obtain another weakly observable offer process.

**Lemma B.1.** *Suppose that the choice function of every hospital is observably substitutable across doctors. Let  $\mathbf{y}$  and  $\mathbf{z}$  be two weakly observable offer processes that are both weakly compatible with the same preference profile  $\succ$ . Then  $(\mathbf{y}, \mathbf{z})$  is a weakly observable offer process.*

*Proof.* Consider any weakly observable offer process  $\mathbf{y} = (y^1, \dots, y^M)$ . We will prove the statement by induction on the length of  $\mathbf{z} = (z^1, \dots, z^N)$ , showing at each step that  $(\mathbf{y}, \mathbf{z})$  and  $(\mathbf{z}, \mathbf{y})$  are weakly observable. If  $N = 0$ , the statement is trivially true. Hence, suppose that  $(\mathbf{y}, (z^1, \dots, z^{N-1}))$  and  $((z^1, \dots, z^{N-1}), \mathbf{y})$  are weakly observable.

We first show that  $(\mathbf{y}, \mathbf{z})$  is weakly observable. There are two cases:

1. If  $z^N \in \mathbf{c}(\mathbf{y})$ , then  $(\mathbf{y}, (z^1, \dots, z^{N-1})) = (\mathbf{y}, \mathbf{z})$  and so  $(\mathbf{y}, \mathbf{z})$  is weakly observable by the inductive assumption.
2. If  $z^N \notin \mathbf{c}(\mathbf{y})$ , we first note that  $(\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}(\mathbf{z})) \cap (X_{\mathbf{d}(z^N)} \cap X_{\mathbf{h}(z^N)}) = \emptyset$ ,<sup>52</sup> that is, no contract between  $\mathbf{d}(z^N)$  and  $\mathbf{h}(z^N)$  is in offer process  $\mathbf{y}$  unless it is also in  $(z^1, \dots, z^{N-1})$ . Since  $\mathbf{z}$  is weakly observable, we must have  $\mathbf{d}(z^N) \notin \mathbf{d}(C^{\mathbf{h}(z^N)}(\{z^1, \dots, z^{N-1}\}))$ . By the inductive assumption,  $((z^1, \dots, z^{N-1}), \mathbf{y})$  is weakly observable. Since  $C^{\mathbf{h}(z^N)}$  is observably substitutable across doctors, we then obtain that  $\mathbf{d}(z^N) \notin \mathbf{d}(C^{\mathbf{h}(z^N)}(\{z^1, \dots, z^{N-1}\} \cup \mathbf{c}(\mathbf{y})))$  given that  $(\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}(\mathbf{z})) \cap (X_{\mathbf{d}(z^N)} \cap X_{\mathbf{h}(z^N)}) = \emptyset$ ; therefore,  $(\mathbf{y}, \mathbf{z})$  is weakly observable by definition.

We now show by induction on  $m$  that, for all  $m \leq M$ , the offer process  $(\mathbf{z}, (y^1, \dots, y^m))$  is weakly observable. Suppose that for some  $\bar{m} \leq M - 1$ , the statement has already been shown for all  $m' \leq \bar{m}$ . We will show that the statement holds for  $\bar{m} + 1$ . There are two cases:

1. If  $y^{\bar{m}+1} \in \mathbf{c}(\mathbf{z})$ , then  $(\mathbf{z}, (y^1, \dots, y^{\bar{m}+1})) = (\mathbf{z}, (y^1, \dots, y^{\bar{m}}))$  and  $(\mathbf{z}, (y^1, \dots, y^{\bar{m}+1}))$  is weakly observable by the inductive assumption.
2. If  $y^{\bar{m}+1} \notin \mathbf{c}(\mathbf{z})$ , we first note that  $(\mathbf{c}(\mathbf{z}) \setminus \mathbf{c}(\mathbf{y})) \cap (X_{\mathbf{d}(y^{\bar{m}+1})} \cap X_{\mathbf{h}(y^{\bar{m}+1})}) = \emptyset$ ,<sup>53</sup> that is, no contract between  $\mathbf{d}(y^{\bar{m}+1})$  and  $\mathbf{h}(y^{\bar{m}+1})$  is in offer process  $\mathbf{z}$  unless it was also in  $\mathbf{y}$ . Since  $\mathbf{y}$  is weakly observable, we must have  $\mathbf{d}(y^{\bar{m}+1}) \notin \mathbf{d}(C^{\mathbf{h}(y^{\bar{m}+1})}(\{y^1, \dots, y^{\bar{m}}\}))$ . We have already established that  $((y^1, \dots, y^{\bar{m}}), \mathbf{z})$  is weakly observable. Since  $C^{\mathbf{h}(y^{\bar{m}+1})}$  is observably substitutable across doctors, we then obtain that  $\mathbf{d}(y^{\bar{m}+1}) \notin \mathbf{d}(C^{\mathbf{h}(y^{\bar{m}+1})}(\{y^1, \dots, y^{\bar{m}}\} \cup \mathbf{c}(\mathbf{z})))$  given that  $(\mathbf{c}(\mathbf{z}) \setminus \mathbf{c}(\mathbf{y})) \cap (X_{\mathbf{d}(y^{\bar{m}+1})} \cap X_{\mathbf{h}(y^{\bar{m}+1})}) = \emptyset$ ; therefore,  $(\mathbf{z}, (y^1, \dots, y^{\bar{m}+1}))$  is weakly observable by definition.

<sup>52</sup>Since  $z^N \notin \mathbf{c}(\mathbf{y})$ , we have that for all  $z \in \mathbf{c}(\mathbf{y}) \cap (X_{\mathbf{d}(z^N)} \cap X_{\mathbf{h}(z^N)})$  it must be the case that  $z \succ_{\mathbf{d}(z^N)} z^N$ . Hence, if there existed  $w \in (\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}(\mathbf{z})) \cap (X_{\mathbf{d}(z^N)} \cap X_{\mathbf{h}(z^N)})$ , then  $\mathbf{z}$  and  $\mathbf{y}$  would not be weakly compatible with the same preference profile.

<sup>53</sup>Since  $y^{\bar{m}+1} \notin \mathbf{c}(\mathbf{z})$ , we have that for all  $z \in \mathbf{c}(\mathbf{y}) \cap (X_{\mathbf{d}(y^{\bar{m}+1})} \cap X_{\mathbf{h}(y^{\bar{m}+1})})$  it must be the case that  $z \succ_{\mathbf{d}(y^{\bar{m}+1})} y^{\bar{m}+1}$ . Hence, if there existed  $w \in (\mathbf{c}(\mathbf{z}) \setminus \mathbf{c}(\mathbf{y})) \cap (X_{\mathbf{d}(y^{\bar{m}+1})} \cap X_{\mathbf{h}(y^{\bar{m}+1})})$ , then  $\mathbf{z}$  and  $\mathbf{y}$  would not be weakly compatible with the same preference profile.

This completes the proof of Lemma B.1. □

Our second preliminary lemma shows that

- if the choice function of each hospital is observably substitutable, then the set of rejected contracts expands monotonically along combined offer processes that are weakly observable, and
- if the choice function of each hospital is observably size monotonic, then the set of chosen contracts grows weakly larger along combined offer processes that are weakly observable.

**Lemma B.2.** *Let  $\mathbf{y}$  and  $\mathbf{z}$  be two offer processes such that  $(\mathbf{y}, \mathbf{z})$  is a weakly observable offer process:*

- *If the choice function of each hospital is observably substitutable, then  $R^H(\mathbf{c}(\mathbf{y})) \subseteq R^H(\mathbf{c}(\mathbf{y}) \cup \mathbf{c}(\mathbf{z}))$ .*
- *If the choice function of each hospital is observably substitutable across doctors,  $d \notin \mathbf{d}(C^H(\mathbf{c}(\mathbf{y})))$  and  $d \notin \mathbf{d}(\mathbf{c}(\mathbf{z}) \setminus \mathbf{c}(\mathbf{y}))$  implies  $d \notin \mathbf{d}(C^H(\mathbf{c}(\mathbf{y}) \cup \mathbf{c}(\mathbf{z})))$ .*
- *If the choice function of each hospital is observably size monotonic, then  $|C^H(\mathbf{c}(\mathbf{y}))| \leq |C^H(\mathbf{c}(\mathbf{y}) \cup \mathbf{c}(\mathbf{z}))|$ .*

*Proof.* Define  $\ell_k$  for  $k \geq 1$  inductively as  $\ell_k \equiv \min\{\ell \in \{1, \dots, N\} : z^\ell \notin \mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}\}$ .

The proof is by induction on  $k$ .

- We first show the result for observable substitutability: Suppose that we have already established that  $R^H(\mathbf{c}(\mathbf{y})) \subseteq R^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\})$ ; we will show that  $R^H(\mathbf{c}(\mathbf{y})) \subseteq R^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})$ . Since  $(\mathbf{y}, \mathbf{z})$  is a weakly observable offer process,  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  is a weakly observable offer process. Let  $\mathbf{w} = (w^1, \dots, w^M)$  be the offer process for  $\mathbf{h}(z^{\ell_k})$  constructed by taking the subsequence of  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  that consists of contracts with  $\mathbf{h}(z^{\ell_k})$ ; since  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  is weakly observable, we have that  $\mathbf{w}$  is observable. As the choice function of  $C^{\mathbf{h}(z^{\ell_k})}$  is observably substitutable, we have that  $R^{\mathbf{h}(z^{\ell_k})}(\{w^1, \dots, w^{M-1}\}) \subseteq R^{\mathbf{h}(z^{\ell_k})}(\mathbf{w})$ . Moreover, since  $[\{w^1, \dots, w^{M-1}\}]_h = [\mathbf{c}(\mathbf{w})]_h$  for all  $h \in H \setminus \{\mathbf{h}(z^{\ell_k})\}$ , we have that  $R^h(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}) = R^h(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})$  for all  $h \in H \setminus \{\mathbf{h}(z^{\ell_k})\}$ . Combining the preceding two observations, we obtain that

$$R^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}) \subseteq R^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\});$$

combining this with our inductive hypothesis that  $R^H(\mathbf{c}(\mathbf{y})) \subseteq R^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\})$ , we obtain the desired result that  $R^H(\mathbf{c}(\mathbf{y})) \subseteq R^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})$ .

- Next, we show the result for observable substitutability across doctors: Suppose that we have already established that  $d \notin d(C^H(\mathbf{c}(\mathbf{y})))$  and  $d \notin d(\{\{z^{\ell_1}, \dots, z^{\ell_{k-1}}\} \setminus \mathbf{c}(\mathbf{y})\})$  imply  $d \notin d(C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}))$ ; we will show that  $d \neq d(z^{\ell_k})$  implies that  $d \notin d(C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\}))$ . For any hospital  $h \neq h(z^{\ell_k})$ , since  $[\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}]_h = [\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\}]_h$  and  $d \notin d(C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}))$ , we have that  $d \notin d(C^h(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\}))$ . For  $h = h(z^{\ell_k})$ , since  $(\mathbf{y}, \mathbf{z})$  is a weakly observable offer process, the offer process  $\mathbf{w} = (w^1, \dots, w^M)$  constructed by taking the subsequence of  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  that consists of contracts with  $h(z^{\ell_k})$  is observable; note that  $w^M = z^{\ell_k}$ . But then, since

$$d \notin d(C^{h(w^M)}(\{w^1, \dots, w^{M-1}\})) = d(C^{h(w^M)}(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\})),$$

$d \neq d(w^M) = d(z^{\ell_k})$ , and the choice function of  $h(w^M)$  is observably substitutable across doctors, we have  $d \notin d(C^{h(w^M)}(\{w^1, \dots, w^M\}))$ . Thus,  $d \notin d(C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\}))$ .

- We now show the result for observable size monotonicity: Suppose that we have already established that  $|C^H(\mathbf{c}(\mathbf{y}))| \leq |C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\})|$ ; we will show that  $|C^H(\mathbf{c}(\mathbf{y}))| \leq |C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})|$ . Since  $(\mathbf{y}, \mathbf{z})$  is a weakly observable offer process,  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  is a weakly observable process. Let  $\mathbf{w} = (w^1, \dots, w^M)$  be the offer process for  $h(z^{\ell_k})$  constructed by taking the subsequence of  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  that consists of contracts with  $h(z^{\ell_k})$ ; since  $(\mathbf{y}, z^{\ell_1}, \dots, z^{\ell_k})$  is weakly observable, we have that  $\mathbf{w}$  is observable. As the choice function of  $C^{h(z^{\ell_k})}$  is observably size monotonic, we have that  $|C^{h(z^{\ell_k})}(\{w^1, \dots, w^{M-1}\})| \leq |C^{h(z^{\ell_k})}(\mathbf{w})|$ . Moreover, since  $[\{w^1, \dots, w^{M-1}\}]_h = [\mathbf{c}(\mathbf{w})]_h$  for all  $h \in H \setminus \{h(z^{\ell_k})\}$ , we have that  $C^h(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\}) = C^h(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})$  for all  $h \in H \setminus \{h(z^{\ell_k})\}$ . Combining the preceding two observations, we obtain that

$$|C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\})| \leq |C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})|;$$

combining this with our inductive hypothesis that  $|C^H(\mathbf{c}(\mathbf{y}))| \leq |C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_{k-1}}\})|$ , we obtain the desired result that  $|C^H(\mathbf{c}(\mathbf{y}))| \leq |C^H(\mathbf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \dots, z^{\ell_k}\})|$ .

□

Our third preliminary lemma uses Proposition 3 to show that the outcome of the cumulative offer mechanism under a preference profile  $\succ$ , i.e.,  $\mathcal{C}(\succ)$ , is equal to the choice of the hospitals from the contracts in an offer process  $\mathbf{x}$  that is complete with respect to  $\succ$ , i.e.,  $C^H(\mathbf{c}(\mathbf{x}))$ .

**Lemma B.3.** *Suppose that the choice function of every hospital is observably substitutable across doctors. If  $\mathbf{x}$  is a complete offer process with respect to  $\succ$ , then  $\mathcal{C}(\succ) = C^H(\mathbf{c}(\mathbf{x}))$ .*

*Proof.* As shown in the proof of Proposition 3, any complete offer process with respect to a given preference profile  $\succ$  has the same set of contracts. Moreover, a complete offer process  $\mathbf{x} = (x^1, \dots, x^K)$  corresponds to a cumulative offer mechanism with an ordering  $\vdash$  such that  $x^k \vdash x^{k'}$  if  $k < k'$ , in the sense that the set of available contracts at the end of the cumulative offer mechanism with respect to  $\succ$  and  $\vdash$  is  $\mathbf{c}(\mathbf{x})$ , as we can calculate that  $x^k$  is proposed in the  $k^{\text{th}}$  step of the cumulative offer mechanism with respect to  $\succ$  and  $\vdash$ . Thus,  $\mathcal{C}(\succ) = C^H(\mathbf{c}(\mathbf{x}))$ .  $\square$

Our fourth and final preliminary lemma shows that cumulative offer mechanisms are truncation-consistent when choice functions are observably substitutable.

**Lemma B.4.** *Suppose that the choice function of every hospital is observably substitutable. For any ordering  $\vdash$  of  $X$ ,  $\mathcal{C}^\vdash$  is truncation-consistent.*

*Proof.* Let  $\succ$  be an arbitrary preference profile,  $d$  be an arbitrary doctor, and  $\mathbf{x} = (x^1, \dots, x^T)$  be the sequence of contracts that are proposed in the cumulative offer mechanism for  $\vdash$  and  $\succ$ . For any  $h \in H$ , let  $\mathbf{x}^h$  be the subsequence of  $\mathbf{x}$  that contains all contracts that name hospital  $h$ . Since the rules of cumulative offer mechanisms dictate that a doctor proposes a new contract only when all previously proposed contracts have been rejected, it is immediate that  $\mathbf{x}^h$  is an observable offer process for any  $h \in H$ . Hence, observable substitutability implies that  $R^H(\{x^1, \dots, x^t\}) \subseteq R^H(\{x^1, \dots, x^{t+1}\})$  for all  $t \leq T - 1$ . We will use the last observation repeatedly in our proof.

Assume first that  $\mathcal{C}_d^\vdash(\succ) = x \succ_d \emptyset$  and let  $\hat{\succ}_d$  be an arbitrary truncation of  $\succ_d$  under which  $x$  is acceptable. Observable substitutability implies that there cannot be a Step  $t \leq T$  of the cumulative offer mechanism for  $\vdash$  and  $\succ$  at which  $x$  is rejected by  $\mathbf{h}(x)$ . But then, the cumulative offer mechanism for  $\vdash$  and  $(\hat{\succ}_d, \succ_{D \setminus \{d\}})$  produces exactly the same sequence of proposed contracts—and therefore also the same outcome—as the cumulative offer mechanism for  $\vdash$  and  $\succ$ . In particular, we obtain  $\mathcal{C}_d^\vdash(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = x$ .

Next, assume that  $\mathcal{C}_d^\vdash(\succ) = \emptyset$  and let  $\hat{\succ}_d$  be a preference relation for  $d$  such that  $\succ_d$  is a truncation of  $\hat{\succ}_d$ . Since  $d$  is unemployed under  $\mathcal{C}_d^\vdash(\succ)$ , there must be some earliest step  $T^* \leq T$  of the cumulative offer process such that  $R^H(\{x^1, \dots, x^{T^*}\})$  contains all contracts that are acceptable with respect to  $\succ_d$ . Now let  $\mathbf{y} = (y^1, \dots, y^{T'})$  be the sequence of contracts proposed in the cumulative offer mechanism for  $\vdash$  and  $(\hat{\succ}_d, \succ_{D \setminus \{d\}})$ . Since  $\succ_d$  is a truncation of  $\hat{\succ}_d$ , we must have that  $y^t = x^t$  for all  $t \leq T^*$ . Hence,  $R^H(\{y^1, \dots, y^{T^*}\})$  contains all acceptable contracts with respect to  $\succ_d$ . By observable substitutability,  $R^H(\mathbf{c}(\mathbf{y}))$  must

then also contain all acceptable contracts with respect to  $\succ_d$ ; this observation completes the proof.  $\square$

## B.1 Proof of Theorem 1a

Recall that truncation-consistency is a weaker property than strategy-proofness. Hence, Theorem B.1 implies Theorem 1a.

**Theorem B.1.** *If  $|H| > 1$  and the choice function of some hospital is not observably substitutable, then there exist unit-demand choice functions for the other hospitals such that no stable and truncation-consistent mechanism exists.*

For the proof of Theorem B.1, it is useful to introduce an alternative definition of observable substitutability that operates on sets of contracts.

**Definition B.1.** A set  $Y$  is *observably substitutable* under the choice profile  $C = (C^h)_{h \in H}$  if, for any observable offer process  $\mathbf{x} = (x^1, \dots, x^M)$  such that  $\mathbf{c}(\mathbf{x}) \subseteq Y$ , we have that  $R^H(\{x^1, \dots, x^{M-1}\}) \subseteq R^H(\{x^1, \dots, x^M\})$ .

Note that a choice function  $C^h$  is *observably substitutable* according to Definition 2 if, and only if,  $X_h$  is observably substitutable under  $C^h$  according to Definition B.1. Furthermore, note that if  $Y \subseteq X$  is observably substitutable under  $C = (C^h)_{h \in H}$ , then any  $Z \subseteq Y$  is also observably substitutable under  $(C^h)_{h \in H}$ .

It will also be helpful to define the *lower contour set of an offer process*  $\mathbf{y} = (y^1, \dots, y^K)$ ,

$$\mathbf{L}(\mathbf{y}) \equiv \{y^k \in \mathbf{c}(\mathbf{y}) : \nexists \hat{k} > k \text{ such that } \mathbf{d}(y^k) = \mathbf{d}(y^{\hat{k}})\};$$

that is,  $\mathbf{L}(\mathbf{y})$  contains, for each doctor  $d \in \mathbf{d}(\mathbf{c}(\mathbf{y}))$ , the last contract in  $\mathbf{y}$  that  $d$  is associated with.

The proof of Theorem B.1 will rely on the following lemma, which we prove first.

**Lemma B.5.** *Suppose that the mechanism  $\mathcal{M}$  is stable and truncation-consistent. Suppose that  $Y \subseteq X$  is observably substitutable. Let  $\succ$  be an arbitrary profile of preferences that is consistent with  $Y$ . If  $\mathbf{y}$  is a complete offer process with respect to  $\succ$ , then  $\mathcal{M}(\succ) = C^H(\mathbf{c}(\mathbf{y}))$  and  $C^H(\mathbf{c}(\mathbf{y})) \subseteq \mathbf{L}(\mathbf{y})$ .*

*Proof.* We proceed by induction on  $M \equiv |Y|$ . Our full inductive hypothesis is that for every preference profile  $\succ$  consistent with  $Y$ , for any complete offer process  $\mathbf{y}$  with respect to  $\succ$ ,

1.  $\mathcal{M}(\succ) = C^H(\mathbf{c}(\mathbf{y}))$ , and

2.  $\mathcal{M}(\succ) \subseteq \mathbf{L}(\mathbf{y})$ .

The inductive hypothesis is clearly true for  $M = 0$ , that is, when  $Y = \emptyset$ . Now suppose it is true for all observably substitutable sets of size  $M$  or less. Now consider a set  $Y$  such that  $|Y| = M + 1$ . Consider any preference profile  $\succ$  consistent with  $Y$  and any complete offer process  $\mathbf{y} = (y^1, \dots, y^N)$  with respect to  $\succ$ .

**Observation B.1.** *For each doctor  $d$ , we have that either  $\mathcal{M}_d(\succ) = [\mathbf{L}(\mathbf{y})]_d$  or  $\mathcal{M}_d(\succ) = \emptyset$ .*

*Proof.* Fix an arbitrary doctor  $d \in D$ . There are two cases:

1. If  $Y_d \setminus \mathbf{c}(\mathbf{y}) \neq \emptyset$ , then note first that  $Y_d \setminus \mathbf{c}(\mathbf{y}) \neq \emptyset$  implies  $[C^H(\mathbf{c}(\mathbf{y}))]_d \neq \emptyset$  since  $\succ$  is consistent with  $Y$  and  $\mathbf{y}$  is complete with respect to  $\succ$ . Furthermore, the assumption that  $Y$  is observably substitutable under  $C = (C^h)_{h \in H}$  implies that  $C^H(\mathbf{c}(\mathbf{y}))$  is a feasible outcome. Hence, there has to exist a unique contract  $y \in [C^H(\mathbf{c}(\mathbf{y}))]_d$ . Now let  $\hat{Y} = Y \setminus (Y_d \setminus \mathbf{c}(\mathbf{y}))$  and  $\hat{\succ} \equiv \succ^{\hat{Y}}$ . Note that  $\hat{\succ}_d$  is a truncation of  $\succ_d$  and that  $\hat{\succ}_{d'} = \succ_{d'}$  for all  $d' \in D \setminus \{d\}$ . Since  $\mathbf{y}$  is a complete offer process with respect to  $\hat{\succ}$  and  $\hat{Y} \subsetneq Y$ , the inductive hypothesis implies that  $\mathcal{M}(\hat{\succ}) = C^H(\mathbf{c}(\mathbf{y}))$  and  $C^H(\mathbf{c}(\mathbf{y})) \subseteq \mathbf{L}(\mathbf{y})$ . In particular,  $\{y\} = \mathcal{M}_d(\hat{\succ})$  and  $y \in \mathbf{L}(\mathbf{y})$ . If  $\mathcal{M}_d(\succ) \in Y_d \setminus \mathbf{c}(\mathbf{y})$ , then, since  $\mathbf{y}$  is a complete offer process with respect to  $\succ$ , we would have that  $\mathcal{M}_d(\hat{\succ}) \succ_d \mathcal{M}_d(\succ)$  and so we would obtain a contradiction to truncation-consistency. Hence, we must have  $\mathcal{M}_d(\succ) \in \mathbf{c}(\mathbf{y})$ . The truncation-consistency of  $\mathcal{M}$  implies  $\mathcal{M}_d(\hat{\succ}) = \mathcal{M}_d(\succ)$ ; combining this last expression with the earlier observations that  $\{y\} = \mathcal{M}_d(\hat{\succ})$  and  $y \in \mathbf{L}(\mathbf{y})$  yields the desired result.
2. If  $Y_d \setminus \mathbf{c}(\mathbf{y}) = \emptyset$ , then since  $\mathcal{M}$  is individually rational,  $\mathcal{M}_d(\succ) \subseteq \mathbf{c}(\mathbf{y})$ . By way of contradiction, suppose that there exists a contract  $\hat{y}$  such that  $\{\hat{y}\} = \mathcal{M}_d(\succ)$  and  $\hat{y} \succ_d [\mathbf{L}(\mathbf{y})]_d$ .<sup>54</sup> Let  $\hat{Y} = \{y \in Y : \mathbf{d}(y) \neq d \text{ or } y \succeq_d \hat{y}\}$  and note that  $|\hat{Y}| < |Y|$  as  $\hat{y} \succ_d [\mathbf{L}(\mathbf{y})]_d$ . Let  $\hat{\succ} \equiv \succ^{\hat{Y}}$ . Note that  $\hat{\succ}_d$  is a truncation of  $\succ_d$  and that  $\hat{\succ}_{D \setminus \{d\}} = \succ_{D \setminus \{d\}}$ . As  $\mathcal{M}$  is truncation-consistent, we obtain  $\mathcal{M}_d(\hat{\succ}) = \mathcal{M}_d(\succ) = \{\hat{y}\}$ . Now, let  $\bar{m} = \min\{m : \hat{y} \in R^H(\{y^1, \dots, y^m\})\}$ .<sup>55</sup> Construct a complete offer process  $\mathbf{x} = (x^1, \dots, x^{\bar{N}})$  with respect to  $\hat{\succ}$  such that  $x^n = y^n$  for all  $n = 1, \dots, \bar{m}$ . Since  $\hat{\succ}$  is consistent with  $\hat{Y}$  and  $|\hat{Y}| < |Y|$ , the inductive assumption implies  $\mathcal{M}(\hat{\succ}) \subseteq \mathbf{L}(\mathbf{x})$  and  $\mathcal{M}(\hat{\succ}) = C^H(\mathbf{c}(\mathbf{x}))$ . Since the set  $Y$  is observably substitutable under  $\{C^h\}_{h \in H}$ , we must have  $\hat{y} \in R^H(\{x^1, \dots, x^{\bar{N}}\})$ . Therefore, we must have that  $\hat{y} \notin C^H(\mathbf{c}(\mathbf{x})) = \mathcal{M}(\hat{\succ})$ , contradicting our earlier conclusion that  $\hat{y} = \mathcal{M}_d(\succ)$ .

<sup>54</sup>Note that, by definition,  $\mathbf{L}(\mathbf{y})$  contains at most one contract with each doctor.

<sup>55</sup>It is clear that such an integer must exist since  $\mathbf{y}$  is compatible with  $\succ_d$  and  $\mathbf{c}(\mathbf{y})$  contains a contract in  $[\mathbf{L}(\mathbf{y})]_d$  that  $d$  likes strictly less than  $\hat{y}$ .

This completes the proof of Observation B.1.  $\square$

Observation B.1 implies that  $\mathcal{M}(\succ) \subseteq \mathbf{L}(\mathbf{y})$ , the latter half of our inductive hypothesis on  $Y$ . We now prove the former half, i.e., that  $\mathcal{M}(\succ) = C^H(\mathbf{c}(\mathbf{y}))$ . Suppose it is the case that  $\mathcal{M}(\succ) \neq C^H(\mathbf{c}(\mathbf{y}))$ . Then there exists a hospital  $h$  such that  $\mathcal{M}_h(\succ) \neq C^h(\mathbf{c}(\mathbf{y}))$ . If  $C^h(\mathbf{c}(\mathbf{y})) \subsetneq \mathcal{M}_h(\succ)$ , then  $\mathcal{M}(\succ)$  is not individually rational for  $h$ . Otherwise, given that each  $d \in \mathbf{d}(C^h(\mathbf{c}(\mathbf{y})) \setminus \mathcal{M}_h(\succ))$  strictly prefers  $[C^h(\mathbf{c}(\mathbf{y}))]_d$  over  $[\mathbf{L}(\mathbf{y})]_d$ ,  $C^h(\mathbf{c}(\mathbf{y})) \setminus \mathcal{M}_h(\succ)$  blocks  $\mathcal{M}(\succ)$ . Hence,  $\mathcal{M}(\succ)$  cannot be stable, a contradiction.  $\square$

With the help of Lemma B.5 we will now prove Theorem B.1. Suppose that the choice function of  $h$  is not observably substitutable. Let  $\mathbf{y} = (y^1, \dots, y^M)$  be an observable offer process such that  $R^h(\{y^1, \dots, y^{M-1}\}) \setminus R^h(\{y^1, \dots, y^M\}) \neq \emptyset$ . Assume without loss of generality that  $\mathbf{y}$  is a *minimal observable violation of substitutability* in the sense that every  $Z \subsetneq \mathbf{c}(\mathbf{y})$  is observably substitutable under the choice profile  $(C^{\hat{h}})_{\hat{h} \in H}$ .

**Claim 1.**  $C^h(\mathbf{c}(\mathbf{y})) \subseteq \mathbf{L}(\mathbf{y})$ .

*Proof.* We show first that, for all preference profiles  $\succ$  consistent with  $(\mathbf{y}, \mathbf{c}(\mathbf{y}))$ ,  $\mathcal{M}(\succ) \subseteq \mathbf{L}(\mathbf{y})$ . Suppose, by way of contradiction, that there exists a preference profile  $\succ$  consistent with  $(\mathbf{y}, \mathbf{c}(\mathbf{y}))$  such that  $\mathcal{M}(\succ) \not\subseteq \mathbf{L}(\mathbf{y})$ . Let  $\hat{y}$  be an arbitrary element of  $\mathcal{M}(\succ) \setminus \mathbf{L}(\mathbf{y})$  and let  $\hat{d} \equiv \mathbf{d}(\hat{y})$ . Note that  $\hat{y} \in \mathcal{M}(\succ) \setminus \mathbf{L}(\mathbf{y})$  implies that there exists a contract  $\tilde{y} \in [\mathbf{L}(\mathbf{y})]_{\hat{d}}$  such that  $\hat{y} \succ_{\hat{d}} \tilde{y}$ . Let  $\hat{Y} = \mathbf{c}(\mathbf{y}) \setminus \{\tilde{y}\}$  and  $\hat{\succ} = \succ^{\hat{Y}}$ . Note that  $\hat{\succ}_{\hat{d}}$  is a truncation of  $\succ_{\hat{d}}$  and that  $\hat{\succ}_{D \setminus \{\hat{d}\}} = \succ_{D \setminus \{\hat{d}\}}$ . Since  $\mathcal{M}$  is truncation-consistent, we obtain  $\hat{y} \in \mathcal{M}(\hat{\succ})$ .

Now, let  $\bar{m} = \min\{m : \hat{y} \in R^H(\{y^1, \dots, y^m\})\}$ ; such an  $\bar{m}$  must exist given that  $\tilde{y} \in \mathbf{c}(\mathbf{y})$  (i.e.,  $\hat{d}$  proposes  $\tilde{y}$  along  $\mathbf{y}$ ) and  $\hat{y} \succ_{\hat{d}} \tilde{y}$ . Let  $\mathbf{x} = (x^1, \dots, x^{\bar{m}})$  be a complete offer process with respect to  $\hat{\succ}$  such that  $x^n = y^n$  for all  $n = 1, \dots, \bar{m}$ . Note that  $\hat{y} \notin C^H(\mathbf{c}(\mathbf{x}))$  since  $\hat{y} \in R^H(\{x^1, \dots, x^{\bar{m}}\})$ ,  $\hat{Y}$  is observably substitutable,<sup>56</sup> and  $\mathbf{x}$  is observable. Moreover, by Lemma B.5,  $\mathcal{M}(\hat{\succ}) = C^H(\mathbf{c}(\mathbf{x}))$ . Hence,  $\hat{y} \notin \mathcal{M}(\hat{\succ})$ , contradicting our earlier conclusion that  $\hat{y} \in \mathcal{M}(\hat{\succ})$ . This shows that we must have  $\mathcal{M}(\succ) \subseteq \mathbf{L}(\mathbf{y})$ .

Now, suppose by way of contradiction that  $C^h(\mathbf{c}(\mathbf{y})) \not\subseteq \mathbf{L}(\mathbf{y})$ . If  $C^h(\mathbf{c}(\mathbf{y})) \not\subseteq \mathbf{L}(\mathbf{y})$ , then  $\mathcal{M}(\succ)$  is blocked by  $C^h(\mathbf{c}(\mathbf{y})) \setminus \mathcal{M}(\succ)$ , contradicting the stability of  $\mathcal{M}$ .  $\square$

To complete the proof of Theorem B.1, we let  $\hat{y} \in R^h(\{y^1, \dots, y^{N-1}\}) \setminus R^h(\{y^1, \dots, y^N\})$  be arbitrary, and note that  $\hat{y} \in R^h(\{y^1, \dots, y^{N-1}\}) \setminus R^h(\{y^1, \dots, y^N\})$  implies  $C^h(\{y^1, \dots, y^N\}) \neq C^h(\{y^1, \dots, y^{N-1}\})$ . As  $C^h$  satisfies the irrelevance of rejected contracts condition, the last statement requires that  $y^N \in C^h(\{y^1, \dots, y^N\})$ . Since  $\mathbf{y}$  is observable and  $y^N \in$

<sup>56</sup>The observable substitutability of  $\hat{Y}$  follows as  $\mathbf{y}$  is a minimal observable violation of substitutability and  $\hat{Y} \subsetneq \mathbf{c}(\mathbf{y})$ .



$C^h(\{y^1, \dots, y^N\})$ , we must have  $\hat{d} \equiv \mathbf{d}(\hat{y}) \neq \mathbf{d}(y^N)$  as no hospital ever chooses two contracts with the same doctor.

We claim that  $\hat{d} \notin \mathbf{d}(C^H(\{y^1, \dots, y^{N-1}\}))$ ; to see this, note first that Claim 1 and  $\hat{y} \in C^H(\mathbf{c}(\mathbf{y}))$  imply that  $\hat{y} \in \mathbf{L}(\mathbf{y})$ . Furthermore, since  $\mathbf{y}$  is a minimal observable violation of substitutability, it has to be the case that  $C^h(\{y^1, \dots, y^{N-1}\}) \subseteq \mathbf{L}((y^1, \dots, y^{N-1}))$ . Since  $\hat{d} \neq \mathbf{d}(y^N)$ , we have that  $[\mathbf{L}((y^1, \dots, y^{N-1}))]_{\hat{d}} = [\mathbf{L}(\mathbf{y})]_{\hat{d}}$  so that  $C^h(\{y^1, \dots, y^{N-1}\}) \cap X_{\hat{d}} \subseteq \{\hat{y}\}$ . Since  $\hat{y} \in R^h(\{y^1, \dots, y^{N-1}\})$ , we obtain the desired statement that  $\hat{d} \notin \mathbf{d}(C^H(\{y^1, \dots, y^{N-1}\}))$ .

Now, let  $h'$  be another hospital, let  $\bar{y}'$  be a contract between  $h'$  and  $\mathbf{d}(y^N) \equiv \bar{d}$ , and let  $\hat{y}'$  be a contract between  $h'$  and  $\hat{d}$ . Let the choice function of  $h'$  be given by

$$C^{h'}(Z) = \begin{cases} \{\hat{y}'\} & \hat{y}' \in Z \\ \{\bar{y}'\} & \hat{y}' \notin Z \text{ and } \bar{y}' \in Z \\ \emptyset & \text{otherwise.} \end{cases}$$

Let  $\succ$  be a preference profile that is consistent with  $(\mathbf{y}, \mathbf{c}(\mathbf{y}) \cup \{\bar{y}'\})$  such that  $y \succ_{\bar{d}} \bar{y}'$  for all  $y \in [\mathbf{c}(\mathbf{y}) \setminus \{y^N\}]_{\bar{d}}$ , and  $\bar{y}' \succ_{\bar{d}} y^N$ . A straightforward variation of the arguments used in the proof of Claim 1 shows that we must have  $\mathcal{M}(\succ) \subseteq \mathbf{L}(\mathbf{y}) \cup \{\bar{y}'\}$ ;<sup>57</sup> the stability of  $\mathcal{M}(\succ)$  then implies that  $\bar{y}' \in \mathcal{M}(\succ)$  and therefore  $y^N \notin \mathcal{M}(\succ)$ . Similarly, the stability of  $\mathcal{M}(\succ)$  implies that  $\mathcal{M}_h(\succ) = C^h(\{y^1, \dots, y^{N-1}\}) \subseteq \mathbf{L}((y^1, \dots, y^{N-1}))$ . Since  $\mathbf{y}$  is a minimal observable violation of substitutability,  $\hat{y} \notin \mathcal{M}_h(\succ)$  and  $\mathcal{M}_{\hat{d}}(\succ) = \emptyset$ .

Now consider a preference profile  $\hat{\succ}$  such that

1.  $\hat{\succ}_{D \setminus \{\hat{d}\}} = \succ_{D \setminus \{\hat{d}\}}$ ,
2. for all  $y, z \in [\mathbf{c}(\mathbf{y})]_{\hat{d}}$ ,  $y \hat{\succ}_{\hat{d}} z$  if and only if  $y \succ_{\hat{d}} z$ , and
3.  $\hat{y}' \hat{\succ}_{\hat{d}} \emptyset$  and, for all  $y \in [\mathbf{c}(\mathbf{y})]_{\hat{d}}$ ,  $y \hat{\succ}_{\hat{d}} \hat{y}'$ .

As  $\mathcal{M}$  is strategy-proof, we must have that either  $\mathcal{M}_{\hat{d}}(\hat{\succ}) = \emptyset$  or  $\mathcal{M}_{\hat{d}}(\hat{\succ}) = \{\hat{y}'\}$ . The stability of  $\mathcal{M}(\hat{\succ})$  then implies that  $\mathcal{M}_{\hat{d}}(\hat{\succ}) = \{\hat{y}'\}$  (since  $\hat{y}'$  will always be chosen by  $h'$ ). Again, a straightforward variation of the arguments used in the proof of Claim 1 shows that we must have  $\mathcal{M}_h(\hat{\succ}) \subseteq \mathbf{L}(\mathbf{y})$ . In particular, each doctor weakly prefers his contract in  $\mathbf{L}(\mathbf{y})$  over his contract in  $\mathcal{M}_h(\hat{\succ})$ . Since at least  $\hat{d}$  strictly prefers  $[\mathbf{L}(\mathbf{y})]_{\hat{d}} = \{\hat{y}\}$  over  $\mathcal{M}_{\hat{d}}(\hat{\succ}) = \{\hat{y}'\}$ , we have that  $\mathcal{M}(\hat{\succ})$  is blocked by  $C^h(\mathbf{c}(\mathbf{y})) \setminus \mathcal{M}(\hat{\succ})$ , contradicting the stability of  $\mathcal{M}$ .

<sup>57</sup>Suppose to the contrary that there exists a contract  $\hat{y} \in \mathcal{M}(\succ) \setminus (\mathbf{L}(\mathbf{y}) \cup \{\bar{y}'\})$ . Let  $\hat{Y} \equiv \{y \in Y : \mathbf{d}(y) \neq \mathbf{d}(\hat{y}) \text{ or } y \succeq_{\mathbf{d}(\hat{y})} \hat{y}\}$  and let  $\mathbf{x}$  be a complete offer process with respect to  $\hat{\succ} \equiv \succ^{\hat{Y}}$ . Lemma B.5 implies that  $\mathcal{M}_h(\hat{\succ}) \subseteq \mathbf{L}(\mathbf{x})$ . Since  $\hat{y} \in \mathcal{M}(\succ) \setminus (\mathbf{L}(\mathbf{y}) \cup \{\bar{y}'\})$ , observability of  $\mathbf{y}$  implies that  $\hat{y} \notin \mathcal{M}(\hat{\succ})$ , a contradiction.

## B.2 Proof of Theorem 2

First, note that we may assume that  $C^h$  is observably substitutable, as otherwise Theorem 1a implies that there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists; thus, we assume that  $C^h$  is observably substitutable.

As the choice function of  $h$  is not observably size monotonic, there exists an observable offer process  $\mathbf{x} = (x^1, \dots, x^M)$  such that  $|C^h(\{x^1, \dots, x^M\})| < |C^h(\{x^1, \dots, x^{M-1}\})|$ . Thus,  $x^M \in C^h(\{x^1, \dots, x^M\})$  (as  $C^h$  satisfies the irrelevance of rejected contracts condition) and so there exist two distinct contracts  $x^p, x^q \in C^h(\{x^1, \dots, x^{M-1}\}) \setminus C^h(\{x^1, \dots, x^M\})$  associated with doctors other than  $\mathbf{d}(x^M)$ ; let  $x = x^M$ ,  $y = x^p$ , and  $z = x^q$ . Let  $\succ$  be the preference profile for the doctors such that

1. if  $i < j$  and  $\mathbf{d}(x^i) = \mathbf{d}(x^j)$ , then  $x^i \succ_{\mathbf{d}(x^i)} x^j$ , and
2.  $w \succ_{\mathbf{d}(w)} \emptyset$  if and only if  $w = x^m$  for some  $m = 1, \dots, M$ ;

that is,  $\mathbf{x}$  is compatible with and complete with respect to  $\succ$ .

Let  $\bar{x}$  be a contract between  $\mathbf{d}(x^M)$  and  $\bar{h} \neq h$ ,  $\bar{y}$  be a contract between  $\mathbf{d}(y)$  and  $\bar{h}$ , and  $\bar{z}$  be a contract between  $\mathbf{d}(z)$  and  $\bar{h}$ , and define  $\hat{\succ}_{\{\mathbf{d}(x), \mathbf{d}(y), \mathbf{d}(z)\}}$  as follows:

1. The doctor  $\mathbf{d}(x)$  finds  $\bar{x}$  acceptable under  $\hat{\succ}_{\mathbf{d}(x)}$  but dispreferable to every contract acceptable under  $\succ_{\mathbf{d}(x)}$ ; that is,  $\hat{\succ}_{\mathbf{d}(x)}^h = \succ_{\mathbf{d}(x)}^h$ ,  $x^m \hat{\succ}_{\mathbf{d}(x)} \bar{x}$  for all  $m = 1, \dots, M$  such that  $\mathbf{d}(x^m) = \mathbf{d}(x)$ ,  $\bar{x} \hat{\succ}_{\mathbf{d}(x)} \emptyset$ , and, for all  $w \in X_{\mathbf{d}(x)} \setminus \{x^1, \dots, x^M, \bar{x}\}$ , we have that  $\emptyset \hat{\succ}_{\mathbf{d}(x)} w$ .
2. The doctor  $\mathbf{d}(y)$  finds  $\bar{y}$  acceptable under  $\hat{\succ}_{\mathbf{d}(y)}$  but dispreferable to every contract acceptable under  $\succ_{\mathbf{d}(y)}$ ; that is,  $\hat{\succ}_{\mathbf{d}(y)}^h = \succ_{\mathbf{d}(y)}^h$ ,  $x^m \hat{\succ}_{\mathbf{d}(y)} \bar{y}$  for all  $m = 1, \dots, M$  such that  $\mathbf{d}(x^m) = \mathbf{d}(y)$ ,  $\bar{y} \hat{\succ}_{\mathbf{d}(y)} \emptyset$ , and, for all  $w \in X_{\mathbf{d}(y)} \setminus \{x^1, \dots, x^M, \bar{y}\}$ , we have that  $\emptyset \hat{\succ}_{\mathbf{d}(y)} w$ .
3. The doctor  $\mathbf{d}(z)$  finds  $\bar{z}$  acceptable under  $\hat{\succ}_{\mathbf{d}(z)}$  but dispreferable to every contract acceptable under  $\succ_{\mathbf{d}(z)}$ ; that is,  $\hat{\succ}_{\mathbf{d}(z)}^h = \succ_{\mathbf{d}(z)}^h$ ,  $x^m \hat{\succ}_{\mathbf{d}(z)} \bar{z}$  for all  $m = 1, \dots, M$  such that  $\mathbf{d}(x^m) = \mathbf{d}(z)$ ,  $\bar{z} \hat{\succ}_{\mathbf{d}(z)} \emptyset$ , and, for all  $w \in X_{\mathbf{d}(z)} \setminus \{x^1, \dots, x^M, \bar{z}\}$ , we have that  $\emptyset \hat{\succ}_{\mathbf{d}(z)} w$ .

Finally, we define the choice function of  $\bar{h}$  as

$$C^{\bar{h}}(Z) = \begin{cases} \{\bar{y}\} & \bar{y} \in Z \\ \{\bar{z}\} & \bar{y} \notin Z \text{ and } \bar{z} \in Z \\ \{\bar{x}\} & \bar{y}, \bar{z} \notin Z \text{ and } \bar{x} \in Z \\ \emptyset & \text{otherwise.} \end{cases}$$

Since  $C^h$  and  $C^{\bar{h}}$  are observably substitutable and since  $\mathcal{M}$  is a stable and strategy-proof mechanism,  $\mathcal{M}$  must produce the same outcome as any cumulative offer mechanism (Theorem 1a); thus we need only consider the cumulative offer mechanism  $\mathcal{M} = \mathcal{C}$ . It is true that

$$\mathcal{C}(\hat{\succ}_{\{d(x),d(y),d(z)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}}) = C^h(\mathbf{c}(\mathbf{x})) \cup \{\bar{y}\}$$

as the offer process  $(\mathbf{x}, \bar{y}, \bar{z})$  is consistent with  $(\hat{\succ}_{\{d(x),d(y),d(z)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$  and produces the outcome  $C^h(\mathbf{c}(\mathbf{x})) \cup \{\bar{y}\}$ ; in particular, we have that  $d(z)$  is unemployed under  $\mathcal{C}(\hat{\succ}_{\{d(x),d(y),d(z)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$ .<sup>58</sup>

Now, let  $\tilde{\succ}_{d(z)}$  be the preference ordering for  $d(z)$  such that  $\bar{z}$  is now  $d(z)$ 's favorite contract; that is,  $\tilde{\succ}_{d(z)}^h = \hat{\succ}_{d(z)}^h = \succ_{d(z)}^h$ ,  $\bar{z} \tilde{\succ}_{d(z)} x^m$  for all  $m = 1, \dots, M$  such that  $d(x^m) = d(z)$ , and, for all  $w \in X_{d(z)} \setminus \{x^1, \dots, x^M, \bar{z}\}$ , we have that  $\emptyset \hat{\succ}_{d(z)} w$ .

If  $\mathcal{C}$  is a strategy-proof mechanism, then we must have that  $d(z)$  is unemployed under the outcome  $\mathcal{C}(\tilde{\succ}_{d(z)}, \hat{\succ}_{\{d(x),d(y)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$ , as otherwise submitting  $\tilde{\succ}_{d(z)}$  when his actual preferences are  $\hat{\succ}_{d(z)}$  and other agents submit  $(\hat{\succ}_{\{d(x),d(y)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$  would be a profitable deviation for  $d(z)$  as he would now be assigned an acceptable (under  $\hat{\succ}_{d(z)}$ ) contract.

Now, let  $\tilde{\succ}_{d(x)}$  be the preference ordering for  $d(x)$  such that  $\bar{x}$  is now preferable to  $x$  for  $d(x)$ ; that is,  $\tilde{\succ}_{d(x)}^h = \hat{\succ}_{d(x)}^h = \succ_{d(x)}^h$ ,  $x^m \tilde{\succ}_{d(x)} \bar{x}$  for all  $m = 1, \dots, M - 1$  such that  $d(x^m) = d(x)$ ,  $\bar{x} \tilde{\succ}_{d(x)} x = x^M$  and, for all  $w \in X_{d(x)} \setminus \{x^1, \dots, x^M, \bar{x}\}$ , we have that  $\emptyset \hat{\succ}_{d(x)} w$ .

It is straightforward that

$$\mathcal{C}(\tilde{\succ}_{\{d(x),d(z)\}}, \hat{\succ}_{d(y)}, \succ_{D \setminus \{d(x),d(y),d(z)\}}) = \mathcal{C}(\tilde{\succ}_{d(z)}, \hat{\succ}_{\{d(x),d(y)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}});$$

in particular,  $d(z)$  is unemployed under  $\mathcal{C}(\tilde{\succ}_{\{d(x),d(z)\}}, \hat{\succ}_{d(y)}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$ .<sup>59</sup>

Finally, consider the preferences  $(\tilde{\succ}_{d(x)}, \hat{\succ}_{\{d(y),d(z)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$ . Then  $(x^1, \dots, x^{M-1}, \bar{x})$  is consistent with and complete with respect to  $(\tilde{\succ}_{d(x)}, \hat{\succ}_{\{d(y),d(z)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$  and so

$$\mathcal{C}(\tilde{\succ}_{d(x)}, \hat{\succ}_{\{d(y),d(z)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}}) = C^h(\{x^1, \dots, x^{M-1}\}) \cup \{\bar{x}\},$$

under which  $d(z)$  is employed at  $z$ . Hence, the cumulative offer mechanism is not strategy-proof.

<sup>58</sup>To see this, consider  $C^{\dagger}$  where, under  $\dagger$ , every contract with  $h$  is ranked before any contract with any other hospital.

<sup>59</sup>To see this, consider  $C^{\dagger}$  where, under  $\dagger$ , the contract with  $\bar{z}$  is ranked first; then the cumulative offer mechanism proceeds exactly as under  $(\tilde{\succ}_{d(z)}, \hat{\succ}_{\{d(x),d(y)\}}, \succ_{D \setminus \{d(x),d(y),d(z)\}})$  except that  $\bar{x}$  is proposed and immediately rejected (as  $\bar{z}$  has already been proposed) at some step.

### B.3 Proof of Theorem 4

The uniqueness claim follows immediately from Theorem 1. The implication “(iii)  $\Rightarrow$  (ii)” is trivial and the implication “(ii)  $\Rightarrow$  (i)” follows from Theorem 1a, Theorem 2, and Theorem 3. Hence, we can complete the proof of Theorem 4 by showing that “(i)  $\Rightarrow$  (iii)”, i.e., by showing that observable substitutability, observable size monotonicity, and non-manipulability via contractual terms are jointly sufficient for the cumulative offer mechanism to be stable and strategy-proof.

By Theorem 6, which does not rely on Theorem 4, observable substitutability of each hospital’s choice function is sufficient for the cumulative offer mechanism to produce a stable outcome. Hence, we only need to establish that if each hospital’s choice function is observably substitutable, observably size monotonic, and non-manipulable via contractual terms, then the cumulative offer mechanism is strategy-proof.<sup>60</sup>

Consider a profile of choice functions  $C = (C^h)_{h \in H}$  such that, for each  $h \in H$ ,  $C^h$  is observably substitutable and observably size monotonic. Suppose that the cumulative offer mechanism is not strategy-proof, so that there exists a preference profile  $\succ$ , a doctor  $\hat{d}$ , and a preference relation  $\hat{\succ}_{\hat{d}}$  such that  $\mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}}) \succ_{\hat{d}} \mathcal{C}(\succ)$ . Let  $\hat{x} \in [\mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}})]_{\hat{d}}$  be the contract that  $\hat{d}$  obtains under  $\hat{\succ} \equiv (\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}})$  and let  $\hat{h} \equiv h(\hat{x})$ . We will show that  $C^{\hat{h}}$  is manipulable via contractual terms.

As a first step of the proof, we introduce several assumptions about the preference profiles  $\succ$  and  $\hat{\succ}$  and show that these assumptions are without loss of generality. Let  $\mathbf{x} = (x^1, \dots, x^K)$  be a complete offer process with respect to  $\succ$  and let  $\hat{\mathbf{x}}$  be a complete offer process with respect to  $\hat{\succ}$ . Note that  $\mathcal{C}(\succ) = C^H(\mathbf{c}(\mathbf{x}))$  and  $\mathcal{C}(\hat{\succ}) = C^H(\mathbf{c}(\hat{\mathbf{x}}))$  by Lemma B.3. By Lemma B.4, it is without loss of generality to assume that (1) all contracts in  $X \setminus (\mathbf{c}(\mathbf{x}) \cup \mathbf{c}(\hat{\mathbf{x}}))$  are unacceptable to the associated doctors under  $\succ$  and  $\hat{\succ}$ , and (2)  $\hat{x}$  is the lowest ranked acceptable contract under  $\succ_{\hat{d}}$  and  $\hat{\succ}_{\hat{d}}$ .<sup>61</sup> Finally, note that by Lemma B.3 we can assume without loss of generality that  $\mathbf{x}$  is the offer process with respect to an ordering  $\vdash$  such that, for all  $x \in X \setminus X_{\hat{d}}$  and all  $y \in X_{\hat{d}}$ ,  $x \vdash y$ . This implies that the cumulative offer mechanism corresponding to  $\mathbf{x}$  ends with the rejection of  $\hat{x}$ , i.e., that  $\hat{x}$  is the unique element

<sup>60</sup>As we show in Online Appendix C.1, irrelevance of rejected contracts is necessary for the stability of the cumulative offer mechanism. Our proof that the cumulative offer mechanism is strategy-proof when each hospital’s choice function is observably substitutable, observably size monotonic, and non-manipulable via contractual terms does not depend on the irrelevance of rejected contracts condition.

<sup>61</sup>For all doctors  $d \in D \setminus \{\hat{d}\}$ , Lemma B.4 implies that truncating  $\succ_d = \hat{\succ}_d$  below  $d$ ’s least preferred contract in  $\mathbf{c}(\mathbf{x}) \cup \mathbf{c}(\hat{\mathbf{x}})$  can neither improve nor worsen  $d$ ’s assignment under  $\succ$  and  $\hat{\succ}$ . Similarly, for doctor  $\hat{d}$ , Lemma B.4 implies that truncating  $\hat{\succ}_{\hat{d}}$  below  $\hat{x}$  can neither improve nor worsen  $\hat{d}$ ’s assignment when others submit preferences according to  $\succ_{D \setminus \{\hat{d}\}}$ ; in either case,  $\hat{d}$  obtains  $\hat{x}$ . Finally, since  $\hat{x} \succ_{\hat{d}} \mathcal{C}(\succ)$ , Lemma B.4 implies that  $\hat{d}$  must remain unassigned when he truncates  $\succ_{\hat{d}}$  below  $\hat{x}$  and others submit preferences according to  $\succ_{D \setminus \{\hat{d}\}}$ .

of  $R^H(\{x^1, \dots, x^K\}) \setminus R^H(\{x^1, \dots, x^{K-1}\})$ .<sup>62</sup>

Now set  $\succ' \equiv \succ^{X_{\hat{h}}}$  and  $\hat{\succ}' \equiv \hat{\succ}^{X_{\hat{h}}}$ . Let  $\mathbf{x}'$  be a complete offer process with respect to  $\succ'$ , and let  $\hat{\mathbf{x}}'$  be a complete offer process with respect to  $\hat{\succ}'$ . By Lemma B.3, we must have that  $\mathcal{C}(\succ') = C^H(c(\mathbf{x}'))$  and  $\mathcal{C}(\hat{\succ}') = C^H(c(\hat{\mathbf{x}}'))$ . To show that the choice function of  $\hat{h}$  is manipulable via contractual terms, it is thus sufficient to establish that  $\mathcal{C}_{\hat{d}}(\succ') = \emptyset$ , i.e.,  $\hat{d}$  does not obtain a contract under  $\succ'$ , and that  $\hat{x} \notin R^{\hat{h}}(c(\hat{\mathbf{x}}'))$ , i.e.,  $\hat{d}$  obtains an acceptable contract under  $\hat{\succ}'$ ; Claim 2 (which is easy) shows the former fact while Claim 3 (which is more difficult) shows the latter.

**Claim 2.** *Doctor  $\hat{d}$  does not obtain a contract under  $\succ'$ , i.e.,  $\mathcal{C}_{\hat{d}}(\succ') = \emptyset$ .*

*Proof.* Let  $(y^1, \dots, y^M)$  be the subsequence of  $\mathbf{x} = (x^1, \dots, x^K)$  that consists of all of the contracts with  $\hat{h}$ . Let  $\bar{m} = \min\{m : \hat{x} \in R^{\hat{h}}(\{y^1, \dots, y^m\})\}$ .<sup>63</sup> Now consider an ordering  $\vdash$  such that  $y^m \vdash y^{m+1}$ , for all  $m \in \{1, \dots, M-1\}$ , and  $y^M \vdash y$ , for all  $y \in X \setminus \{y^1, \dots, y^M\}$ . By the construction of  $\succ'$ , the first  $\bar{m}$  contracts in the complete offer process with respect to  $\succ'$  and  $\vdash$  are  $y^1, \dots, y^{\bar{m}}$ . Given that  $C^{\hat{h}}$  is observably substitutable and  $\hat{x} \in R^{\hat{h}}(\{y^1, \dots, y^{\bar{m}}\})$ ,  $\hat{x}$  must be rejected by  $\hat{h}$  when  $\hat{h}$  has access to all contracts in the complete offer process with respect to  $\succ'$  and  $\vdash$ . By Lemma B.3, this implies  $\hat{x} \in R^{\hat{h}}(c(\mathbf{x}'))$ . Since  $\hat{x}$  is the least-preferred acceptable contract for doctor  $\hat{d}$  under  $\hat{\succ}'$ ; this implies that  $\mathcal{C}_{\hat{d}}(\succ') = \emptyset$ .  $\square$

The remainder of the sufficiency proof of Theorem 4 is devoted to showing Claim 3.

**Claim 3.** *The contract  $\hat{x}$  is not rejected under  $\hat{\succ}'$ , i.e.,  $\hat{x} \notin R^{\hat{h}}(c(\hat{\mathbf{x}}'))$ .*

Before proving Claim 3, we introduce some auxiliary concepts that are useful in the argument. Consider an arbitrary preference profile  $\tilde{\succ}$  and an arbitrary offer process  $\mathbf{z}$ .

**Definition 9.** A *pre-run rejection chain at  $\mathbf{z}$  under  $\tilde{\succ}$*  is a non-empty sequence of contracts  $\mathbf{y} = (y^1, \dots, y^N)$  such that the following conditions are satisfied:

1. For doctor  $d^1 \equiv d(y^1)$ ,
  - (a)  $d^1 \in d(C^H(c(\mathbf{z})))$ ,
  - (b)  $d^1 \notin d(C^{h(y^1)}(c(\mathbf{z})))$ , and
  - (c) for all  $y \in [(X_{h(y^1)} \cap X_{d^1}) \cup \{\emptyset\}] \setminus c(\mathbf{z})$ ,  $y^1 \tilde{\succ}_{d^1} y$ .

<sup>62</sup>To see this, note that, as each hospital's choice function is observably size monotonic, at most one contract is rejected in each step of the cumulative offer mechanism with respect to  $\succ$  and  $\vdash$ . Since  $\hat{x}$  is the least preferred contract with respect to  $\succ_{\hat{d}}$ , the cumulative offer mechanism with respect to  $\succ$  and  $\vdash$  ends as soon as  $\hat{x}$  is rejected.

<sup>63</sup>Such an integer  $\bar{m}$  has to exist as  $\hat{x} \in R^{\hat{h}}(c(\mathbf{x}))$  and  $[c(\mathbf{x})]_{\hat{h}} = \{y^1, \dots, y^M\}$ .

2. For all  $n \in \{2, \dots, N\}$ , for doctor  $d^n \equiv d(y^n)$ ,

(a)  $d^n \neq d^1$ ,

(b)  $d^n \notin d(C^H(c(\mathbf{z}) \cup \{y^1, \dots, y^{n-1}\}))$ ,

(c)  $d^n \in d(R^H(c(\mathbf{z}) \cup \{y^1, \dots, y^{n-1}\}) \setminus R^H(c(\mathbf{z}) \cup \{y^1, \dots, y^{n-2}\}))$ , and

(d)  $y^n \succ_{d^n} y$  for all  $y \in (X_{d^n} \cup \{\emptyset\}) \setminus (c(\mathbf{z}) \cup \{y^1, \dots, y^{n-1}\})$ .

3. We have  $d^1 \in d(R^H(c(\mathbf{z}) \cup c(\mathbf{y})) \setminus R^H(c(\mathbf{z}) \cup \{y^1, \dots, y^{N-1}\}))$ .

Intuitively, a pre-run rejection chain is the vacancy chain that would occur if doctor  $d^1$  were to make the offer  $y^1$  even though  $d^1$  is currently employed.<sup>64</sup> Thus, a pre-run rejection chain is an offer process whose first element is a contract  $y^1$  with a doctor  $d^1$  who is employed by some hospital at the end of  $\mathbf{z}$  (part (a) of Condition 1); the contract  $y^1$  is with a hospital different from the hospital that currently employs  $d^1$  (part (b) of Condition 1); and  $y^1$  is  $d^1$ 's favorite contract at  $h(y^1)$  that has not yet been proposed (part (c) of Condition 1). In each subsequent step  $n$  of the pre-run rejection chain, a doctor  $d^n$  other than  $d^1$  (part (a) of Condition 2) who is not employed by any hospital at the end of  $(\mathbf{z}, y^1, \dots, y^{n-1})$  (part (b) of Condition 2) and, in fact, just had a contract rejected after  $y^{n-1}$  was proposed (part (c) of Condition 2) proposes his favorite contract  $y^n$  that has not yet been proposed (part (d) of Condition 2). The pre-run rejection chain continues until the doctor  $d^1$  has a contract rejected (Condition 3).

A *generalized pre-run rejection chain at  $\mathbf{z}$  under  $\tilde{\succ}$*  is an offer process  $\mathbf{y} = (y^1, \dots, y^L)$  such that for each  $\ell \in \{1, \dots, L\}$ ,  $y^\ell$  is a pre-run rejection chain at  $(\mathbf{z}, y^1, \dots, y^{\ell-1})$  under  $\tilde{\succ}$ . An offer process  $\mathbf{w}$  can be *obtained from  $\mathbf{z}$  by pre-running rejection chains under  $\tilde{\succ}$*  if  $\mathbf{w} = (\mathbf{z}, \mathbf{y})$  for some generalized pre-run rejection chain  $\mathbf{y}$  at  $\mathbf{z}$  under  $\tilde{\succ}$ .

Pre-run rejection chains prove useful in determining whether some  $\hat{x}$ , which was not rejected under  $\hat{\succ}$ , will be rejected under  $\hat{\succ}'$ , i.e., after we remove hospitals other than  $\hat{h}$  from the economy. Pre-running rejection chains after the complete offer process with respect to  $\hat{\succ}_{D \setminus \{\hat{d}\}}$  (where each chain begins with an element of  $c(\hat{\mathbf{x}}') \setminus c(\hat{\mathbf{x}})$ ) allows us to show that the additional proposals to  $\hat{h}$  under  $\hat{\succ}'$  will not induce  $\hat{h}$  to reject  $\hat{x}$ .

If  $\mathbf{z}$  is weakly observable and weakly compatible with  $\tilde{\succ}$ , it follows immediately from the definition of a generalized pre-run rejection chain  $\mathbf{y}$  at  $\mathbf{z}$  that  $(\mathbf{z}, \mathbf{y})$  is weakly observable and weakly compatible with  $\tilde{\succ}$ ; we state this fact as Lemma B.6.

**Lemma B.6.** *If  $\mathbf{z}$  is weakly observable and weakly compatible with  $\tilde{\succ}$  and  $\mathbf{y}$  is a generalized pre-run rejection chain at  $\mathbf{z}$  under  $\tilde{\succ}$ , then  $(\mathbf{z}, \mathbf{y})$  is weakly observable and weakly compatible with  $\tilde{\succ}$ .*

<sup>64</sup>Objects similar to pre-run rejection chains are also used by Dur et al. (2018) and Dworzak (2018).

### B.3.1 Proof of Claim 3

Let  $\check{\mathbf{x}}$  be a complete offer process with respect to  $\succ_{D \setminus \{\hat{d}\}}$ . Note that  $c(\check{\mathbf{x}}) \subseteq (c(\mathbf{x}) \cap c(\hat{\mathbf{x}})) \setminus X_{\hat{d}}$ ; this fact follows from Lemma B.3 since any complete offer process for  $\succ$  or  $\hat{\succ}$  has to contain all contracts that are contained in a complete offer process with respect to an ordering  $\vdash$  such that, for all  $y \in X \setminus X_{\hat{d}}$  and all  $x \in X_{\hat{d}}$ ,  $y \vdash x$ . The key step of our proof lies in the construction of an offer process that can be obtained from  $\check{\mathbf{x}}$  by constructing a generalized pre-run rejection chain from  $\check{\mathbf{x}}$  that satisfies four specific properties.

**Claim 4.** *There exists an offer process  $\mathbf{y}^*$  such that*

1.  $\mathbf{y}^*$  can be obtained from  $\check{\mathbf{x}}$  by pre-running rejection chains under  $\hat{\succ}$ ,
2.  $c(\mathbf{y}^*) \subseteq X \setminus X_{\hat{d}}$ ,
3.  $c(\hat{\mathbf{x}}') \setminus c(\hat{\mathbf{x}}) \subseteq c(\mathbf{y}^*)$ , and
4.  $R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*)) \setminus R^H(c(\hat{\mathbf{x}})) \subseteq R^H(c(\mathbf{y}^*))$ .

Condition 1 ensures in particular that  $\mathbf{y}^*$  is weakly observable (by Lemma B.6). Condition 2 requires that no contract in  $c(\mathbf{y}^*)$  names doctor  $\hat{d}$ . Condition 3 ensures that  $c(\mathbf{y}^*)$  contains all the contracts that are proposed in the cumulative offer mechanism for  $\hat{\succ}'$  that are *not* in the cumulative offer mechanism for  $\hat{\succ}$ . Condition 4 ensures that all the contracts that are rejected when contracts in  $c(\mathbf{y}^*)$  become available to hospitals in addition to contracts in  $c(\hat{\mathbf{x}})$  are contracts that are also rejected when hospitals have access to the contracts in  $c(\mathbf{y}^*)$ .

The proof of Claim 4 is complex and we relegate it to Online Appendix F. Here, we only argue why Claim 4 implies Claim 3, i.e., that  $\hat{x} \notin R^H(c(\hat{\mathbf{x}}'))$ . We take an offer process  $\mathbf{y}^*$  that satisfies the four conditions of Claim 4 and proceed in two steps:

**Observation B.2.** *We have  $\hat{x} \notin R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*))$ .*

*Proof.* To show that  $\hat{x} \notin R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*))$ , note that, by the fourth condition of Claim 4,  $R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*)) \setminus R^H(c(\hat{\mathbf{x}})) \subseteq R^H(c(\mathbf{y}^*))$ . Since  $c(\mathbf{y}^*) \subseteq X \setminus X_{\hat{d}}$  by the second condition of Claim 4, we must have

$$R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*)) \setminus R^H(c(\hat{\mathbf{x}})) \subseteq X \setminus X_{\hat{d}};$$

combining this with the fact that  $\hat{x} \in C^H(c(\hat{\mathbf{x}}))$  (and thus  $\hat{x} \notin R^H(c(\hat{\mathbf{x}}))$ ), it then follows that  $\hat{x} \notin R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*))$ .  $\square$

**Observation B.3.** *We have  $R^H(c(\hat{\mathbf{x}}')) \subseteq R^H(c(\hat{\mathbf{x}}) \cup c(\mathbf{y}^*))$ .*

*Proof.* Since  $\mathbf{y}^*$  can be obtained from  $\check{\mathbf{x}}$  by constructing a generalized pre-run rejection chain at  $\check{\mathbf{x}}$  by the first condition of Claim 4, Lemma B.6 implies that  $\mathbf{y}^*$  is weakly observable and weakly compatible with  $\hat{\succ}$ . Since  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{x}}'$  are also both weakly observable and weakly compatible with  $\hat{\succ}$ ,  $(\hat{\mathbf{x}}', \hat{\mathbf{x}}, \mathbf{y}^*)$  is weakly observable by Lemma B.1. Lemma B.2 then implies that  $R^H(\mathbf{c}(\hat{\mathbf{x}}')) \subseteq R^H(\mathbf{c}(\hat{\mathbf{x}}') \cup \mathbf{c}(\hat{\mathbf{x}}) \cup \mathbf{c}(\mathbf{y}^*))$ . By the third condition of Claim 4,  $\mathbf{c}(\hat{\mathbf{x}}') \setminus \mathbf{c}(\hat{\mathbf{x}}) \subseteq \mathbf{c}(\mathbf{y}^*)$ , and hence  $\mathbf{c}(\hat{\mathbf{x}}') \cup \mathbf{c}(\hat{\mathbf{x}}) \cup \mathbf{c}(\mathbf{y}^*) = \mathbf{c}(\hat{\mathbf{x}}) \cup \mathbf{c}(\mathbf{y}^*)$ . Combining the preceding two findings yields  $R^H(\mathbf{c}(\hat{\mathbf{x}}')) \subseteq R^H(\mathbf{c}(\hat{\mathbf{x}}) \cup \mathbf{c}(\mathbf{y}^*))$ .  $\square$

Combining Observations B.2 and B.3 yields that  $\hat{x} \notin R^H(\mathbf{c}(\hat{\mathbf{x}}'))$ .

## B.4 Proof of Theorem 5

We can restate Lemmata B.1–B.5, Theorem 1, Theorem 6, and Theorem 1a and Theorems 2–4 *mutatis mutandis* for the case in which doctors' preferences are required to be in a subclass  $\mathcal{Q}$ . Moreover, the proofs of these statements also proceed as before *mutatis mutandis*; note that the auxiliary preference profiles constructed in the proofs of Theorem 1a and Theorems 2–4 must be in  $\mathcal{Q}$  since  $\mathcal{Q}$  is a subclass of  $\mathcal{P}$ . Theorem 5 then follows immediately.

## B.5 Proof of Proposition 2

Assume that  $C^h$  is observably substitutable and manipulable via contractual terms (absent other hospitals), and consider a doctor  $d \in D$ , a preference profile  $\succ$  over contracts with  $h$ , and preferences  $\hat{\succ}_d$  for  $d$  over contracts with  $h$  such that  $\mathcal{C}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{C}(\succ)$ . Note that since  $\mathcal{C}(\succ)$  is individually rational and  $\mathcal{C}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{C}(\succ)$ , there has to exist some contract  $\hat{x}$  such that  $\{\hat{x}\} = \mathcal{C}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}})$ .

We argue first that it is without loss of generality to assume that  $\hat{x}$  is the least preferred acceptable contract according to  $\hat{\succ}_d$  and  $\succ_d$ : By Lemma B.4, the cumulative offer mechanism must still assign  $\hat{x}$  to  $d$  when  $d$  truncates  $\hat{\succ}_d$  at  $\hat{x}$  (and everyone else submits preferences according to  $\succ_{D \setminus \{d\}}$ ). Similarly, given that  $\hat{x} \succ_d \mathcal{C}(\succ)$ , we must have  $\hat{x} \succ_d \emptyset$  and Lemma B.4 implies that  $d$  must be unemployed when he truncates  $\succ_d$  at  $\hat{x}$ . Hence, if all other agents submit preferences corresponding to  $\succ_{D \setminus \{d\}}$ , then it is profitable for  $d$  to submit the truncation of  $\hat{\succ}_d$  at  $\hat{x}$  when her true preferences are given by the truncation of  $\succ_d$  at  $\hat{x}$ .

Thus, we can assume that  $\succ_d$  is of the form

$$y^1 \succ_d \dots \succ_d y^{N-1} \succ_d y^N = \hat{x} \succ_d \emptyset$$

and  $\hat{\succ}_d$  is of the form

$$\hat{y}^1 \hat{\succ}_d \dots \hat{\succ}_d \hat{y}^{\hat{N}-1} \hat{\succ}_d \hat{y}^{\hat{N}} = \hat{x} \hat{\succ}_d \emptyset,$$



with  $\mathcal{C}_d(\succ_d, \succ_{D \setminus \{d\}}) = \emptyset$  and  $\mathcal{C}_d(\hat{\succ}_d, \succ_{D \setminus \{d\}}) = \{\hat{x}\}$ .

Let  $\hat{\succ}_d^n$  be given by

$$\hat{y}^n \hat{\succ}_d^n \dots \hat{\succ}_d^n \hat{y}^{\hat{N}} = \hat{x} \hat{\succ}_d^n \emptyset$$

for  $n = 1, \dots, \hat{N}$ .

There are two cases to consider:

**Case 1:**  $\hat{x} \notin \mathcal{C}(\hat{\succ}_d^{\hat{N}}, \succ_{D \setminus \{d\}})$ . Then there exists some  $n \leq \hat{N}$  such that  $\hat{x} \notin \mathcal{C}(\hat{\succ}_d^n, \succ_{D \setminus \{d\}})$  while  $\hat{x} \in \mathcal{C}(\hat{\succ}_d^{n-1}, \succ_{D \setminus \{d\}})$ .

If  $\mathcal{C}(\hat{\succ}_d^{n-1}, \succ_{D \setminus \{d\}}) = \hat{x} \hat{\succ}_d \mathcal{C}(\hat{\succ}_d^n, \succ_{D \setminus \{d\}})$ , then since  $\hat{x}$  is the least preferred acceptable contract according to  $\hat{\succ}_d$ , we have that  $\mathcal{C}_d(\hat{\succ}_d^n, \succ_{D \setminus \{d\}}) = \emptyset$ . Thus,  $\hat{\succ}_d^n$  and  $\hat{\succ}_d^{n-1}$  satisfy the first condition of Proposition 2.

If  $\mathcal{C}(\hat{\succ}_d^n, \succ_{D \setminus \{d\}}) \hat{\succ}_d \hat{x} = \mathcal{C}(\hat{\succ}_d^{n-1}, \succ_{D \setminus \{d\}})$ , then  $\{\hat{y}^m\} = \mathcal{C}_d(\hat{\succ}_d^n, \succ_{D \setminus \{d\}})$  for some  $m < \hat{N}$ . Let  $\tilde{\succ}_d^n$  be given by the truncation of  $\hat{\succ}_d^n$  at  $\hat{y}^m$ , i.e.,

$$\hat{y}^n \tilde{\succ}_d^n \dots \tilde{\succ}_d^n \hat{y}^m \tilde{\succ}_d^n \emptyset;$$

by Lemma B.4, the cumulative offer mechanism must still assign  $\hat{y}^m$  to  $d$  under  $(\tilde{\succ}_d^n, \succ_{D \setminus \{d\}})$ , as  $\tilde{\succ}_d^n$  is a truncation of  $\hat{\succ}_d^n$  at  $\hat{y}^m$ . Similarly, let  $\tilde{\succ}_d^{n-1}$  be given by the truncation of  $\hat{\succ}_d^{n-1}$  at  $\hat{y}^m$ , i.e.,

$$\hat{y}^{n-1} \tilde{\succ}_d^{n-1} \hat{y}^n \tilde{\succ}_d^{n-1} \dots \tilde{\succ}_d^{n-1} \hat{y}^m \tilde{\succ}_d^{n-1} \emptyset;$$

by Lemma B.4, the cumulative offer mechanism must assign  $\emptyset$  to  $d$  under  $(\tilde{\succ}_d^{n-1}, \succ_{D \setminus \{d\}})$ , as  $\tilde{\succ}_d^{n-1}$  is of a truncation of  $\hat{\succ}_d^n$  at  $\hat{y}^m$ . Thus,  $\tilde{\succ}_d^n$  and  $\tilde{\succ}_d^{n-1}$  satisfy the second condition of Proposition 2.

**Case 2:**  $\{\hat{x}\} = \mathcal{C}_d(\hat{\succ}_d^{\hat{N}}, \succ_{D \setminus \{d\}})$ . In this case, let  $\succ_d^n$  be given by

$$y^n \succ_d^n \dots \succ_d^n y^N = \hat{x} \succ_d^n \emptyset$$

for  $n = 1, \dots, N$ . Since  $\succ_d^N = \hat{\succ}_d^{\hat{N}}$  (as under both  $\hat{x}$  is the only contract acceptable to  $d$ ), there must exist some  $n \leq N$  such that  $\{y^m\} = \mathcal{C}_d(\succ_d^n, \succ_{D \setminus \{d\}})$  while simultaneously  $\mathcal{C}_d(\succ_d^{n-1}, \succ_{D \setminus \{d\}}) = \emptyset$  for some  $m \leq N$ . Let  $\tilde{\succ}_d^n$  be given by the truncation of  $\succ_d^n$  at  $y^m$ , i.e.,

$$y^n \tilde{\succ}_d^n \dots \tilde{\succ}_d^n y^m \tilde{\succ}_d^n \emptyset;$$

by Lemma B.4, the cumulative offer mechanism must still assign  $y^m$  to  $d$  under  $(\tilde{\succ}_d^n, \succ_{D \setminus \{d\}})$ , as  $\tilde{\succ}_d^n$  is a truncation of  $\succ_d^n$  at  $y^m$ . Similarly, let  $\tilde{\succ}_d^{n-1}$  be given by the

truncation of  $\succ_d^{n-1}$  at  $y^m$ , i.e.,

$$y^{n-1} \lesssim_d^{n-1} y^n \lesssim_d^{n-1} \dots \lesssim_d^{n-1} y^m \lesssim_d^{n-1} \emptyset;$$

by Lemma B.4, the cumulative offer mechanism must assign  $\emptyset$  to  $d$  under  $(\lesssim_d^{n-1}, \succ_{D \setminus \{d\}})$ , as  $\lesssim_d^{n-1}$  is a truncation of  $\lesssim_d^n$  at  $y^m$ . Thus,  $\lesssim_d^n$  and  $\lesssim_d^{n-1}$  satisfy the second condition of Proposition 2.

## B.6 Proof of Proposition 3

Fix a preference profile  $\succ$ . Let  $\vdash$  be an arbitrary ordering and  $\mathbf{x} = (x^1, \dots, x^M)$  be the corresponding complete offer process, and let  $\vdash'$  be another ordering and  $\mathbf{y} = (y^1, \dots, y^N)$  be the corresponding complete offer process.

We show first that  $\mathbf{c}(\mathbf{x}) \setminus \mathbf{c}(\mathbf{y}) = \emptyset$ . Suppose by way of contradiction that  $\mathbf{c}(\mathbf{x}) \setminus \mathbf{c}(\mathbf{y}) \neq \emptyset$  and let  $m$  be the smallest integer such that  $x^m \notin \mathbf{c}(\mathbf{y})$ . Let  $\mathbf{x}' = (x^1, \dots, x^{m-1})$ . Three facts follow immediately:

1.  $d(x^m) \notin d(C^H(\mathbf{c}(\mathbf{x}')))$ , as  $\mathbf{x}$  is an observable offer process.
2.  $d(x^m) \in d(C^H(\mathbf{c}(\mathbf{y})))$ , as  $x^m \succ_{d(x^m)} \emptyset$ ,  $x^m \notin \mathbf{c}(\mathbf{y})$ , and  $\mathbf{y}$  is a complete offer process.
3.  $d(x^m) \notin d(\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}(\mathbf{x}'))$ , as  $\mathbf{c}(\mathbf{y}) \cap X_{d(x^m)} \subseteq \mathbf{c}(\mathbf{x}')$  since  $x^m \notin \mathbf{c}(\mathbf{y})$ , each  $x \in X_{d(x^m)}$  such that  $x \succ_{d(x^m)} x^m$  is in  $\mathbf{c}(\mathbf{x}')$  (as  $\mathbf{x}'$  is an offer process), and  $\mathbf{y}$  is a complete offer process.

Now, since  $\mathbf{x}'$  and  $\mathbf{y}$  are both compatible with respect to the same preference profile  $\succ$ , we can apply Lemma B.1 to infer that  $(\mathbf{x}', \mathbf{y})$  is weakly observable. Since  $C^h$  is observably substitutable across doctors for all  $h \in H$ , we must have that, since  $d(x^m) \notin C^H(\mathbf{c}(\mathbf{x}'))$  (Fact 1) and  $d(x^m) \notin d(\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}(\mathbf{x}'))$  (Fact 3), then

$$d(x^m) \notin C^H(\mathbf{c}(\mathbf{x}') \cup \mathbf{c}(\mathbf{y})) = C^H(\mathbf{c}(\mathbf{y})), \quad (1)$$

where the last equality follows from the fact that  $\mathbf{c}(\mathbf{x}') \subseteq \mathbf{c}(\mathbf{y})$  by construction. But (1) contradicts Fact 2.

The proof that  $\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}(\mathbf{x}) = \emptyset$  is analogous.

## B.7 Proof of Theorem 6

Fix a profile of choice functions  $C = (C^h)_{h \in H}$  that is observably substitutable across doctors, a preference profile  $\succ = (\succ_d)_{d \in D}$  for the doctors, and an ordering  $\vdash$  of the elements of  $X$ .

For any  $t \geq 1$ , let  $y^t$  denote the (unique) contract that is offered in Step  $t$  of the cumulative offer mechanism with respect to  $\vdash$  and  $\succ$  and set  $A^t \equiv \{y^1, \dots, y^t\}$ .

We first show by induction on  $t$  that  $C^H(A^t)$  is a feasible outcome. For  $t = 0$ , there is nothing to show. So suppose the statement is true up to some  $t \geq 0$  and consider Step  $t+1$ . Let  $h^{t+1} \equiv h(y^{t+1})$ . Note that for any  $h \neq h^{t+1}$ , we have that  $A_h^t = A_h^{t+1}$  and  $C^h(A^t) = C^h(A^{t+1})$ . Suppose there exists a contract  $x \in C^{h^{t+1}}(A^{t+1}) \setminus \{y^{t+1}\}$ ; observable substitutability across doctors then implies that  $\mathbf{d}(x) \in \mathbf{d}(C^{h^{t+1}}(A^t))$ . Hence,  $x \in C^{h^{t+1}}(A^{t+1}) \setminus \{y^{t+1}\}$  and the inductive assumption imply that  $\mathbf{d}(x) \notin \mathbf{d}(C^h(A^t)) = \mathbf{d}(C^h(A^{t+1}))$ , for all  $h \neq h^{t+1}$ . This shows that  $C^H(A^{t+1})$  is a feasible outcome.

Next, we show that  $A \equiv C^H(A^T)$  is stable. By construction,  $A$  is individually rational for hospitals. Moreover, each doctor only proposes acceptable contracts. To see that  $A$  is unblocked, consider an arbitrary set of contracts  $Z \subseteq X \setminus A$  such that  $Z \succ_d A$  for all  $d \in \mathbf{d}(Z)$ . As every doctor proposes during the cumulative offer mechanism every contract preferable to their assigned contract, we must have  $Z \subseteq A^T \setminus A$ . Since  $A = C^H(A^T)$  and  $Z \subseteq X \setminus A$ , irrelevance of rejected contracts implies  $A = C^H(A \cup Z)$ .<sup>65</sup> Hence,  $Z$  is not a blocking set of  $A$ .

## B.8 Proof of Theorem 7

Let  $h \in H$  be an arbitrary hospital and assume that  $C^h$  is not observably substitutable across doctors. Let  $\mathbf{x} = (x^1, \dots, x^M)$  be an observable offer process for  $h$  for which there exists a contract  $x \in \{x^1, \dots, x^{M-1}\}$  such that  $x \in C^h(\mathbf{c}(\mathbf{x}))$  even though  $\mathbf{d}(x) \notin \mathbf{d}(C^h(\{x^1, \dots, x^{M-1}\}))$ . Assume without loss of generality that  $\mathbf{x}$  is *minimal* in the sense that, for all observable offer processes  $\mathbf{y} = (y^1, \dots, y^N)$  such that  $\mathbf{c}(\mathbf{y}) \subsetneq \mathbf{c}(\mathbf{x})$ ,  $y \in R^h(\{y^1, \dots, y^{N-1}\}) \setminus R^h(\{y^1, \dots, y^N\})$  implies that  $\mathbf{d}(y) \in \mathbf{d}(C^h(\{y^1, \dots, y^{M-1}\}))$ .

Let  $\bar{x}$  be a contract between  $\mathbf{d}(x)$  and a hospital  $\bar{h} \neq h$  and  $\bar{x}^M$  be a contract between  $\mathbf{d}(x^M)$  and  $\bar{h}$ .

For the doctors, we define  $\succ$  as follows:

1. For all  $m, m'$  such that  $m < m'$  and  $\mathbf{d}(x^m) = \mathbf{d}(x^{m'})$ , we set  $x^m \succ_{\mathbf{d}(x^m)} x^{m'} \succ_{\mathbf{d}(x^m)} \emptyset$ .
2. We set  $\bar{x} \succ_{\mathbf{d}(x)} \emptyset$  and, for all  $m \in \{1, \dots, M-1\}$  such that  $\mathbf{d}(x^m) = \mathbf{d}(x)$ , we set  $x^m \succ_{\mathbf{d}(x)} \bar{x}$ .
3. We set  $\bar{x}^M \succ_{\mathbf{d}(x^M)} x^M \succ_{\mathbf{d}(x^M)} \emptyset$  and, for all  $m \in \{1, \dots, M-1\}$  such that  $\mathbf{d}(x^m) = \mathbf{d}(x^M)$ , we set  $x^m \succ_{\mathbf{d}(x^M)} \bar{x}^M$ .

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<sup>65</sup>Example C.1 in Online Appendix C.1 shows that the irrelevance of rejected contracts condition is necessary to guarantee the existence of stable outcomes even when the choice functions of hospitals are observably substitutable and observably size monotonic.

For  $\bar{h}$ , we set

$$C^{\bar{h}}(Y) = \begin{cases} \{\bar{x}\} & \bar{x} \in Y \\ \{\bar{x}^M\} & \bar{x} \notin Y \text{ and } \bar{x}^M \in Y \\ \emptyset & \text{otherwise.} \end{cases}$$

We show that for any ordering  $\vdash$ , the set of contracts proposed in the cumulative offer mechanism with respect to  $\succ$  and  $\vdash$  must be  $\mathbf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$ ; this will be sufficient to prove Theorem 7 since  $C^H(\mathbf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}) = C^h(\{x^1, \dots, x^M\}) \cup \{\bar{x}\}$  and  $\mathbf{d}(\bar{x}) = \mathbf{d}(x) \in \mathbf{d}(C^h(\{x^1, \dots, x^M\}))$ , so that the outcome of any cumulative offer mechanism for  $\succ$ , i.e.,  $C^H(\mathbf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\})$ , is not even feasible, as a contract with  $\mathbf{d}(x)$  is chosen by both  $h$  and  $\bar{h}$ .

For the remainder of the proof, fix an arbitrary ordering  $\vdash$  and let  $\mathbf{y}$  be the complete offer process that is produced by the cumulative offer mechanism with respect to  $\succ$  and  $\vdash$ . Note that we must have  $\mathbf{c}(\mathbf{y}) \subseteq \mathbf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$ , since by construction doctors only find contracts in  $\mathbf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$  acceptable. Now suppose by way of contradiction that there is an  $m$  such that  $x^m \notin \mathbf{c}(\mathbf{y})$ ; take  $m$  to be the smallest such integer, so that  $\{x^1, \dots, x^{m-1}\} \subseteq \mathbf{c}(\mathbf{y})$ . Since  $x^m \notin \mathbf{c}(\mathbf{y})$ ,  $\mathbf{y}$  is a complete offer process, and  $x^m \succ_{\mathbf{d}(x^m)} \emptyset$ , we must have  $\mathbf{d}(x^m) \in \mathbf{d}(C^H(\mathbf{c}(\mathbf{y})))$ .

Since  $(x^1, \dots, x^{m-1})$  and  $\mathbf{y}$  are (weakly) observable offer processes, Lemma B.1 implies that the offer process  $\mathbf{z} = ((x^1, \dots, x^{m-1}), \mathbf{y})$  is weakly observable. Since  $\mathbf{x}$  is observable, we have that

$$\mathbf{d}(x^m) \notin \mathbf{d}(C^h(\{x^1, \dots, x^{m-1}\})). \quad (2)$$

Since  $\mathbf{y}$  is a complete offer process and  $x^m \notin \mathbf{c}(\mathbf{y})$ , the compatibility of  $\mathbf{y}$  with  $\succ$  implies that  $\{x^m, \dots, x^M\}_{\mathbf{d}(x^m)} \cap \mathbf{c}(\mathbf{y}) = \emptyset$ , and so no contract with  $\mathbf{d}(x^m)$  is proposed in  $\mathbf{z} = (x^1, \dots, x^{m-1}, \mathbf{y})$  after  $x^{m-1}$ ; in particular, we have that

$$\mathbf{d}(x^m) \notin \mathbf{d}([\mathbf{c}(\mathbf{y}) \setminus \mathbf{c}((x^1, \dots, x^{m-1}))]_h). \quad (3)$$

Combining (2) and (3) with the second part of Lemma B.2,<sup>66</sup> we obtain that  $\mathbf{d}(x^m) \notin \mathbf{d}(C^h(\mathbf{c}(\mathbf{z})))$  and, since  $\mathbf{c}(\mathbf{z}) = \mathbf{c}(\mathbf{y})$ , we have that  $\mathbf{d}(x^m) \notin \mathbf{d}(C^h(\mathbf{c}(\mathbf{y})))$ , contradicting our conclusion in the prior paragraph that  $\mathbf{d}(x^m) \in \mathbf{d}(C^h(\mathbf{c}(\mathbf{y})))$ .

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<sup>66</sup>Note that the minimality of  $\mathbf{x}$  implies that the choice function of  $h$  is observably substitutable across doctors for the offer processes  $(x^1, \dots, x^{m-1})$  and  $\mathbf{y}$ .