IRRATIONAL ROOTS REVISITED

SCOTT DUKE KOMINERS

Estermann [1] gave a clever proof of the irrationality of $\sqrt{2}$, by comparing $\sqrt{2}$ to $\lfloor\sqrt{2}\rfloor$. Hughes [2] extended Estermann's method to show that, for any positive integer n, if $\sqrt{n}$ is rational then n is a perfect square.

In fact, Estermann's argument may be extended further. Here, we generalize Hughes's approach to show that, for positive integers n and k, $\sqrt[k]{n}$ is rational only when n is a perfect k-th power.

Proposition. *Suppose that n and k are positive integers such that $\sqrt[k]{n}$ is rational. Then, n is a perfect k-th power.*

Proof. Suppose that $\sqrt[k]{n} = \frac{a}{b}$, where $a, b \geq 0$ and $(a,b) = 1$. Then, we let

$$c = b\left(\sqrt[k]{n} - \lfloor\sqrt[k]{n}\rfloor\right).$$

We observe that $0 \leq c < b$, since $0 \leq \sqrt[k]{n} - \lfloor\sqrt[k]{n}\rfloor < 1$.

But then,

$$\begin{aligned} c\left(\frac{a}{b}\right)^{k-1} &= c(\sqrt[k]{n})^{k-1} = b(\sqrt[k]{n})^{k} - b\lfloor\sqrt[k]{n}\rfloor(\sqrt[k]{n})^{k-1} \\ &= bn - b\lfloor\sqrt[k]{n}\rfloor\left(\frac{a}{b}\right)^{k-1}. \qquad (1) \end{aligned}$$

Multiplying both sides of (1) by b^{k-2} gives

$$\frac{ca^{k-1}}{b} = b^{k-1}n - \lfloor\sqrt[k]{n}\rfloor a^{k-1} \in \mathbb{Z}. \qquad (2)$$

But we see from (2) that $b \mid ca^{k-1}$. Since $(a,b) = 1$, it follows that $b \mid c$. However, $0 \leq c < b$ by construction, hence we must have $c = 0$. Then, $\sqrt[k]{n} = \lfloor\sqrt[k]{n}\rfloor$—$n$ is a perfect k-th power. □

References

[1] T. Estermann, The irrationality of $\sqrt{2}$, *Math. Gaz.* **59** (June 1975) p. 110.

[2] Colin Richard Hughes, Irrational roots, *Math. Gaz.* **83** (November 1999) pp. 502–503.

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