Lone Wolves in Competitive Equilibria*

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Abstract

This paper develop a class of equilibrium-independent predictions of competitive equilibrium with indivisibilities. Specifically, we prove an analogue of the "Lone Wolf Theorem" of classical matching theory, showing that when utility is perfectly transferable, any agent who does not participate in trade in one competitive equilibrium must receive his autarky payoff in every competitive equilibrium. Our results extend to approximate equilibria and to settings in which utility is only approximately transferable.

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1 Introduction

In models of exchange and production, results on the uniqueness of equilibria typically require continuity and convexity conditions on agents' preferences, in addition to further conditions on aggregate demand (see, for example, Section 17.F of Mas-Colell et al. (1995)). However, continuity and convexity conditions fail in the presence of indivisibilities, leading to the existence of multiple equilibria. In turn, the presence of multiple equilibria weakens the predictive power of economic models.

To avoid the problems of multiplicity, we often consider predictions that do not depend on which equilibrium is realized (e.g., set-identification—see, e.g., Galichon and Henry (2011); Bontemps and Magnac (2017)). In this paper, we develop a class of equilibrium-independent predictions of competitive equilibrium in exchange economies with indivisibilities.[1] Specifically, we show that when utility is perfectly transferable, any agent who does not participate in trade in one competitive equilibrium must receive his autarky payoff in every competitive equilibrium.

Our result is analogous to the classical "Lone Wolf Theorem" in matching theory, which asserts that in one-to-one matching without transfers, any agent who is unmatched in one stable outcome is unmatched in every stable outcome (McVitie and Wilson (1970)). Informally, the Lone Wolf Theorem shows the set of "lone wolves" is equilibrium-independent. Since our setting involves continuous transfers, and hence indifferences, we cannot obtain sharp predictions regarding the set of agents that participate in trade in equilibrium. Instead, we show that any agent who does not participate in trade in one equilibrium cannot benefit from trade in any equilibrium.

When utility is perfectly transferable, there is generically a unique efficient allocation of goods among agents, although there may be multiple possible equilibrium price vectors. Hence, the set of agents that participate in trade is generically equilibrium-independent. However,

[1]We use the Baldwin and Klemperer (2015) model, which the nests the models of Gul and Stacchetti (1999), Sun and Yang (2006), as well as the Hatfield et al. (2013) model of matching with trading networks with transferable utility.

our lone wolf result applies even when there are multiple efficient allocations. Moreover, we show that versions of our Lone Wolf Theorem hold in settings where multiple allocations can be robustly sustained in equilibrium. Specifically, we consider an approximate equilibrium concept—which may be a reasonable solution when the economy suffers from small frictions—and derive an approximate Lone Wolf Theorem. We also prove an approximate Lone Wolf Theorem for equilibria for economies in which utility is only approximately transferable among agents.

Quasi-conversely, we show that some assumption on the transferability of utility is essential to our lone wolf results. Specifically, we show by example that lone wolf results do not hold in settings with strong income effects. Hence, while our lone wolf results show that competitive equilibrium has some (approximately) robust predictions when utility is (approximately) transferable, it remains an open question to derive specific classes of equilibrium-independent predictions more generally.

In a companion paper (Jagadeesan et al. (2017)), we use our Lone Wolf Theorem to show that descending salary-adjustment processes are strategy-proof in quasilinear Kelso and Crawford (1982) economies. Our argument for strategy-proofness rests crucially on applying our Lone Wolf Theorem to (non-generic) economies with multiple efficient matchings.

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 presents our lone wolf results. Section 4 discusses the role of the hypothesis that utility is (approximately) transferable. Section 5 connects our results to the matching literature. Proofs ommitted from the main text are presented in Appendix A.

2 Model

We consider a general model of exchange economies with indivisible goods and transferable utility, following Baldwin and Klemperer (2015); this framework embeds the trading network model of Hatfield et al. (2013), which in turn generalizes the settings of Gul and Stacchetti

(1999, 2000) and Sun and Yang (2006, 2009). There is a set Γ of goods and a finite set I of agents.

Each agent has an endowment $e^i \in \mathbb{Z}^\Gamma$ and a valuation

$$v^i : \mathbb{Z}^\Gamma \to \mathbb{R} \cup \{-\infty\}.$$

We assume that valuations are bounded above; however, we do not impose any further conditions on them. The valuation v^i induces a *quasilinear* utility function u^i by

$$u^i(q^i; t^i) = v^i(q^i) - t^i.$$

A competitive equilibrium is comprised of (1) an allocation of goods to agents and (2) prices for each good such that the allocation maximizes each agent's utility given prices.

Definition 1. A *competitive equilibrium* is a pair $[q; p]$ at which

- each agent i demands his allocation q^i at prevailing prices p, i.e.,

$$q^i \in \arg\max_{\hat{q}^i \in \mathbb{Z}^\Gamma} \left\{ u^i(\hat{q}^i; (e^i - \hat{q}^i) \cdot p) \right\} \tag{1}$$

  for all $i \in I$, and

- the market clears, i.e.,

$$\sum_{i \in I} q^i = \sum_{i \in I} e^i. \tag{2}$$

Remark. Although our model formally requires that goods are indivisible, our results apply in settings with divisible goods as well.

3 Results

The classical Lone Wolf Theorem from the theory of two-sided matching without transfers asserts that any agent who is matched in some equilibrium outcome is matched in every equilibrium (McVitie and Wilson (1970); see also Roth (1984, 1986)). In this section, we prove analogues of the Lone Wolf Theorem for exchange economies.

3.1 An Exact Lone Wolf Theorem

For a pair $[q;p]$, we say that an agent $i \in I$ *does not participate in trade in* $[q;p]$ if $q^i = e^i$. Such an agent is a "lone wolf" in $[q;p]$.

Our first result asserts that any agent who does not participate in trade in some competitive equilibrium receives his autarky payoff in every competitive equilibrium. Thus, we show that any agent who is a "lone wolf" in some equilibrium cannot benefit from trade in *any* equilibrium.

Theorem 1. *Let* $[q;p]$ *and* $[\hat{q};\hat{p}]$ *be competitive equilibria, and let* $j \in I$ *be an agent. If* j *does not participate in trade in* $[\hat{q};\hat{p}]$ *(i.e., if* $\hat{q}^j = e^j$*), then* $u^j(q^j;t^j) = u^j(e^j;0)$*, where* $t^j = (e^j - q^j) \cdot p$*.*

The key to the proof of Theorem 1 is the following lemma, which allows us to produce a new competitive equilibrium from any two competitive equilibria.

Lemma 1 (Shapley (1964); Hatfield et al. (2013))**.** *If* $[q;p]$ *and* $[\hat{q};\hat{p}]$ *are competitive equilibria, then* $[\hat{q};p]$ *is a competitive equilibrium.*

To prove Theorem 1, we exploit the fact that, as $[\hat{q};p]$ is a competitive equilibrium (by Lemma 1), the allocation $\hat{q}$ maximizes j's utility given the price vector p.

Proof of Theorem 1. By Lemma 1, $[\hat{q};p]$ is a competitive equilibrium. Since both $[q;p]$ and $[\hat{q};p]$ are competitive equilibria, every agent i must be indifferent between consumption bundles q^i and $\hat{q}^i$ at price vector p. In particular, j must be indifferent between consumption bundles q^j and $e^j = \hat{q}^j$ at price vector p. Formally, we have that

$$u^j(q^j; t^j) = u^j(q^j; (e^j - q^j) \cdot p) = u^j(e^j; (e^i - e^j) \cdot p) = u^j(e^j; 0),$$

as claimed. $\square$

3.2 An Approximate Lone Wolf Theorem

Note that Theorem 1 applies only in economies in which there are multiple efficient allocations. As efficient allocations are generically unique in transferable utility economies, Theorem 1 has nontrivial consequences only for nongeneric economies.

However, as we show in this section, our lone wolf result holds more generally. In particular, we prove a version of Theorem 1 for approximate equilibria; as approximate equilibria are only approximately efficient, this result has nontrivial consequences in settings in which equilibrium allocations are robustly non-unique.

We relax the notion of competitive equilibrium by allowing agents' total maximization error to be positive but bounded above by ε.

Definition 2. An ε-*equilibrium* consists of a pair $[q; p]$ for which

$$\sum_{i \in I} \left(\max_{\hat{q}^i \in \mathbb{Z}^\Gamma} u^i(\hat{q}^i; (e^i - \hat{q}^i) \cdot p) - u^i(q^i; (e^i - q^i) \cdot p) \right) \leq \varepsilon$$

and (2) is satisfied (i.e., the market clears).

Our Approximate Lone Wolf Theorem asserts that if there exists an ε-equilibrium in which no agents in $J \subseteq I$ participate in trade, then the difference between the total utility of agents in J and the total autarky payoff of agents in J is bounded above by $(\delta + \varepsilon)$ in every δ-equilibrium.

Theorem 2. *Let $[q;p]$ be a δ-equilibrium, let $[\hat{q};\hat{p}]$ be an ε-equilibrium, and let $J \subseteq I$ be a set of agents. If no agent in J participates in trade in $[\hat{q};\hat{p}]$ (i.e., if $\hat{q}^j = e^j$ for all $j \in J$), then*

$$\sum_{j\in J} u^j(q^j;t^j) - \sum_{j\in J} u^j(e^j;0) \leq \delta + \varepsilon,$$

where $t^j = (e^j - q^j) \cdot p$.

The proof of Theorem 2 is similar to the proof of Theorem 1, but relies on an approximate version of Lemma 1.

Taking $J = \{j\}$ yields the following corollary, which corresponds more directly to Theorem 1.

Corollary 1. *Let $[q;p]$ be a δ-equilibrium, let $[\hat{q};\hat{p}]$ be an ε-equilibrium, and let $j \in I$ be an agent. If j does not participate in trade in $[\hat{q};\hat{p}]$ (i.e., if $\hat{q}^j = e^j$), then*

$$u^j(q^j;t^j) - u^j(e^j;0) \leq \delta + \varepsilon,$$

where $t^j = (e^j - q^j) \cdot p$.

3.3 A Lone Wolf Theorem for Economies with Approximately Transferable Utility

Theorem 2 implies a Lone Wolf Theorem for economies in which utility is "close to transferable" in a formal sense. Specifically, we consider economies in which agents' utility functions can be well-approximated by quasilinear utility functions.

Definition 3. A utility function u^i is *quasilinear within* η if there exists a quasilinear utility function $\check{u}^i$ for which

$$\sup_{q^i,t^i} \left\{ \left| u^i(q^i;t^i) - \check{u}^i(q^i;t^i) \right| \right\} \leq \eta.$$

When utility functions are approximately quasilinear, we obtain an approximate Lone Wolf Theorem.

Theorem 3. *Let $[q;p]$ and $[\hat{q};\hat{p}]$ be competitive equilibria, and let $J \subseteq I$ be a set of agents. If no agent in J participates in trade in $[\hat{q};\hat{p}]$ (i.e., if $\hat{q}^j = e^j$ for all $j \in J$) and u^i is quasilinear within η for all $i \in I$, then*

$$\sum_{j\in J} u^j(q^j;t^j) - \sum_{j\in J} u^j(e^j;0) \leq 6\eta|I|,$$

where $t^j = (e^j - q^j)\cdot p$.

To prove Theorem 3, we construct an approximating economy in which utility functions are quasilinear. Competitive equilibria in the original economy give rise to approximate equilibria in the approximating economy; Theorem 3 then follows from Theorem 2.

Proof of Theorem 3. We define an auxiliary economy in which utility functions are given by $(\breve{u}^i)_{i\in I}$, with each $\breve{u}^i$ quasilinear as in Definition 3. By construction, $[q;p]$ and $[\hat{q};\hat{p}]$ are $2\eta|I|$-equilibria in the auxiliary economy. Theorem 2 then guarantees that

$$\sum_{j\in J} \breve{u}^j(q^j;t^j) - \sum_{j\in J} \breve{u}^j(e^j;0) \leq 4\eta|I|. \tag{3}$$

As $\sum_{j\in J} |u^j(q^j;t^j) - \breve{u}^j(q^j;t^j)| \leq \eta|I|$ and $\sum_{j\in J} |u^j(e^j;0) - \breve{u}^j(e^j;0)| \leq \eta|I|$ by our choice of $(\breve{u}^i)_{i\in I}$, we see from (3) that

$$\sum_{j\in J} u^j(q^j;t^j) - \sum_{j\in J} u^j(e^j;0) \leq 6\eta|I|,$$

as desired. □

4 The Role of Utility Transferability

The proofs of our Lone Wolf Theorems use the assumption that utility is (at least approximately) transferable. In fact, the Lone Wolf Theorem fails when utility is not transferable, such as in the presence of income effects.

Consider an economy with four agents, a house builder Build, two real estate agents HighEnd and LowEnd, and a consumer Consume. Real estate agent HighEnd specializes in high-end properties, while LowEnd specializes in low-end properties. The builder Build can construct a high-end property or a low-end property and sell it to the consumer Consume via HighEnd or LowEnd, respectively. These possible interactions can be summarized in the following network:[2]

$$\begin{tikzcd} & \mathsf{Build} \arrow[dl] \arrow[dr] & \\ \mathsf{HighEnd} \arrow[dr] & & \mathsf{LowEnd}. \arrow[dl] \\ & \mathsf{Consume} & \end{tikzcd} \tag{4}$$

It costs Build 110 to construct a high-end property and 50 to construct a low-end property. It costs each real estate agent 0 to intermediate between Build and Consume. We suppose that Consume values the high-end property more than the low-end property but experiences income effects—when prices are high, Consume prefers to buy the low-end property. Intuitively, the high-end property might require additional fees, such as maintenance costs and property taxes, which could cause Consume to prefer the low-end property when prices are high, but not when prices are low. Moreover, Consume cannot costlessly convert a high-end property into a low-end property.[3] Formally, Consume has utility function

$$u^{\mathsf{Consume}}(\text{high-end property}, t) = 2t + 600$$

$$u^{\mathsf{Consume}}(\text{low-end property}, t) = t + 400.$$

As there is only one real estate agent of each type (or, more generally, since we do not assume free entry in the markets for real estate agents), it may be possible for the real estate agents to extract rents. Thus, a competitive equilibrium must specify prices that the

[2]Formally, the trading network is a case of the model of Fleiner et al. (2017), who incorporate income effects (and frictions) into the transferable-utility trading network model of Hatfield et al. (2013).

[3]We rule out intentional-flooding and dynamite-based solutions by assumption.

consumer faces for each type of property separately from the prices that real estate agents face.

There are two possible equilibrium allocations: either Build can construct a high-end property and sell it to Consume via HighEnd, or Build can construct a low-end property and sell it to Consume via LowEnd. For example, two competitive equilibria are:

$$\begin{tikzcd} & \mathsf{Build} \arrow[dl, "\boxed{115}"'] \arrow[dr, "50"] & \\ \mathsf{HighEnd} \arrow[dr, "\boxed{120}"'] & & \mathsf{LowEnd} \arrow[dl, "50"] \\ & \mathsf{Consume} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathsf{Build} \arrow[dl, "152"'] \arrow[dr, "\boxed{95}"] & \\ \mathsf{HighEnd} \arrow[dr, "152"'] & & \mathsf{LowEnd}. \arrow[dl, "\boxed{100}"] \\ & \mathsf{Consume} & \end{tikzcd}$$

(A) (B)

(Here, we denote competitive equilibria by writing a price for each interaction in (4) and boxing the prices of interactions that occur.) In Equilibrium (A), HighEnd extracts a rent of 5 from intermediating between Build and Consume; in Equilibrium (B), LowEnd extracts a rent of 5 from intermediating. However, HighEnd does not trade in Equilibrium (B), while LowEnd does not trade in Equilibrium (A). Thus, there are agents who do not trade in one equilibrium but receive utility strictly greater than their autarky payoffs in the other equilibrium—that is, the lone wolf result fails.

In our example, the builder is able to extract more surplus when the low-end property is traded because the consumer is willing to pay more for the low-end property due to the failure of free disposal. Similarly, the consumer is able to extract more surplus from the builder when the high-end property due to having higher marginal utility of wealth after buying the high-end property. Therefore, the high-end real estate agent is able to improve the consumers' utility, while the low-end real estate agent improves the builders utility, making it possible for either real estate agent to extract rents.[4]

In contrast, our Lone Wolf Theorem shows that if utility is transferable, then at most one of the real estate agents can extract rents. Intuitively, if utility is transferable, then the builder and consumer agree on which possible trade is better.[5] In the presence of income effects, on the other hand, agents can contribute to the social surplus in one allocation but not in another. For example, HighEnd must contribute to the economy when the high-end property is traded, as he is able to extract rents in Equilibrium (A). However, because HighEnd does not participate in trade in Equilibrium (B), he cannot possibly contribute to the economy when the low-end property is traded.

5 Discussion and Conclusion

We developed a class of Lone Wolf Theorems that provide equilibrium-independent predictions of competitive equilibrium analysis in contexts with indivisibilities. Our results show that when utility is perfectly transferable, any agent who does not participate in trade in one competitive equilibrium must receive his autarky payoff in every competitive equilibrium; moreover, this result holds approximately under approximate solution concepts and in settings in which utility is only approximately transferable. Partial transferability of utility is essential

[4] Formally, the extremal equilibria are

$$\begin{tikzcd} & \mathsf{Build} \arrow[dl, "\boxed{110}"'] \arrow[dr, "-20"] & \\ \mathsf{HighEnd} \arrow[dr, "\boxed{110}"'] & & \mathsf{LowEnd} \arrow[dl, "-20"] \\ & \mathsf{Consume} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathsf{Build} \arrow[dl, "460"'] \arrow[dr, "\boxed{400}"] & \\ \mathsf{HighEnd} \arrow[dr, "460"'] & & \mathsf{LowEnd}. \arrow[dl, "\boxed{400}"] \\ & \mathsf{Consume} & \end{tikzcd}$$

(C) (D)

Equilibrium (C) is the equilibrium with lowest prices (and is in particular buyer-optimal), while equilibrium (D) is the equilibrium with highest prices (and is in particular seller-optimal). Note that the high-end property is traded in (C) while the low-end property is traded in (D).

[5] It might be the case that both trades generate the same social surplus—but when utility is transferable, the builder and the consumer will agree on this fact.

for our results—the lone wolf conclusion fails when utility is not transferable (e.g., in the presence of income effects).

5.1 Relationship to the Lone Wolf Theorems and Rural Hospitals Theorems of Matching Theory

Our lone wolf results extend a classical matching-theoretic insight of McVitie and Wilson (1970) to exchange economies. Other analogues of the Lone Wolf Theorem have been developed in matching, but those results have different hypotheses and conclusions from ours.

The most-general matching-theoretic generalizations—developed by Hatfield and Kominers (2012) and Fleiner, Jankó, Tamura, and Teytelboym (2015)—state that for every agent in a trading network (without transfers), the difference between the numbers of the goods bought and sold is invariant across stable outcomes.[6] Our results extend the Lone Wolf Theorem to exchange economies (with transfers) by assessing *when* agents participate in (profitable) trade instead of analyzing the *amounts* that agents trade. Furthermore, the matching-theoretic results rely on two regularity conditions—some form of (gross) substitutability (Kelso and Crawford, 1982), and the law of aggregate demand[7]—which we do not require. We instead require that utility is at least approximately transferable.

Recently, Schlegel (2016) has proved a lone wolf result for many-to-one matching with continuous transfers. While Schlegel (2016) allows workers (i.e., agents on the unit-demand side) to experience income effects, he requires that firms (i.e., agents on the multiunit-demand side) have utility functions that are not only quasilinear but also grossly substitutable. Thus, the Schlegel (2016) lone wolf result is logically independent of ours.

[6]These results are typically called *Rural Hospitals Theorems*, because they imply that that clearinghouses cannot improve the recruitment total of rural hospitals—which are often less attractive to doctors—by switching to different (stable) market-clearing mechanisms. The first Rural Hospitals Theorems were proven by Roth (1984, 1986) for many-to-one matching with responsive preferences. Rural Hospitals Theorems were also developed for many-to-many matching (Alkan (2002); Klijn and Yazıcı (2014)), many-to-one matching with contracts (Hatfield and Milgrom (2005); Hatfield and Kojima (2010)), and many-to-many with contracts (Hatfield and Kominers (2017)).

[7]The law of aggregate demand follows from substitutability in settings with continuous transfers and quasilinear utility functions (Hatfield and Milgrom (2005); Hatfield et al. (2017)).

5.2 Application to Strategy-Proofness

It has been well known since the work of Dubins and Freedman (1981) and Roth (1982) that the Gale–Shapley (1962) *deferred acceptance* mechanism is dominant-strategy incentive compatible for all unit-demand agents on the "proposing" side of the market; this *one-sided strategy-proofness* result has been important in practice (see, e.g., Roth (1984); Balinski and Sönmez (1999); Roth and Peranson (1999); Abdulkadiroğlu and Sönmez (2003); Abdulkadiroğlu, Pathak, and Roth (2005); Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005); Pathak and Sönmez (2008, 2013)) and has been extended to nearly all settings with discrete transfers (or other discrete contracts; see Roth and Sotomayor (1990); Hatfield and Milgrom (2005); Hatfield and Kojima (2009, 2010); Hatfield and Kominers (2012, 2015); Hatfield, Kominers, and Westkamp (2016)).

One-sided strategy-proofness results have heretofore been difficult to derive in matching settings with continuous transfers—in part because the now-standard proof of one-sided strategy-proofness (due to Hatfield and Milgrom (2005)) relies on the matching-theoretic Lone Wolf Theorem, and prior to our work there was no lone wolf result for settings with transfers.[8] In a companion paper (Jagadeesan et al. (2017)), we use the results developed here to give the first direct proof of one-sided strategy-proofness for worker–firm matching under continuously transferable utility.[9]

[8]Indeed, it appears that until Hatfield, Kojima, and Kominers (2017) found an indirect argument by way of a meta-theorem on *ex ante* investment, one-sided strategy-proofness of deferred acceptance in the presence of continuously transferable utility was known only for one-to-one markets (Demange (1982); Leonard (1983); Demange and Gale (1985); Demange (1987)).

[9]We work in the quasilinear case of the Crawford and Knoer (1981) and Kelso and Crawford (1982) models.

A Proofs Omitted from the Main Text

A.1 Proof of Lemma 1

As competitive equilibria are efficient, we have

$$\sum_{i \in I} v^i(q^i) = \max_{\bar{q}^i \in \mathbb{Z}^\Gamma} \left\{ \sum_{i \in I} v^i(\bar{q}^i) \right\} = \sum_{i \in I} v^i(\hat{q}^i).$$

As $\sum_{i \in I} q^i = \sum_{i \in I} e^i = \sum_{i \in I} \hat{q}^i$, we see that

$$\begin{aligned}
\sum_{i \in I} u^i(q^i; (e^i - q^i) \cdot p) &= \sum_{i \in I} v^i(q^i) + \sum_{i \in I} \left(e^i - q^i\right) \cdot p \\
&= \sum_{i \in I} v^i(q^i) + \left(\sum_{i \in I} e^i - \sum_{i \in I} q^i \right) \cdot p \\
&= \sum_{i \in I} v^i(q^i) \\
&= \sum_{i \in I} v^i(\hat{q}^i) \\
&= \sum_{i \in I} v^i(\hat{q}^i) + \left(\sum_{i \in I} e^i - \sum_{i \in I} \hat{q}^i \right) \cdot p \\
&= \sum_{i \in I} v^i(\hat{q}^i) + \sum_{i \in I} \left(e^i - \hat{q}^i\right) \cdot p \\
&= \sum_{i \in I} u^i(\hat{q}^i; (e^i - \hat{q}^i) \cdot p). \qquad (5)
\end{aligned}$$

As $[q; p]$ is a competitive equilibrium, we have

$$q^i \in \arg\max_{\bar{q}^i \in \mathbb{Z}^\Gamma} \left\{ u^i(\bar{q}^i; \left(e^i - \bar{q}^i\right) \cdot p) \right\},$$

for all $i \in I$. Thus, we must have

$$\hat{q}^i \in \arg\max_{\bar{q}^i \in \mathbb{Z}^\Gamma} \left\{ u^i(\bar{q}^i; (e^i - \bar{q}^i) \cdot p) \right\}$$

for all $i \in I$, or else (5) could not be an equality. Hence, we see that $[\hat{q};p]$ is a competitive equilibrium, as claimed.

A.2 Proof of Theorem 2

We first prove an approximate version of Lemma 1.

Lemma 2. *If $[q;p]$ is a δ-equilibrium and $[\hat{q};\hat{p}]$ is an ε-equilibrium, then $[\hat{q};p]$ is a $(\delta+\varepsilon)$-equilibrium.*

Proof. As $[\hat{q};\hat{p}]$ is an ε-equilibrium, it must be within ε of efficiency. In particular, we must have

$$\sum_{i\in I} v^i(\hat{q}^i) + \varepsilon \geq \sum_{i\in I} v^i(q^i).$$

Since $\sum_{i\in I} q^i = \sum_{i\in I} e^i = \sum_{i\in I} \hat{q}^i$, it follows that

$$\begin{aligned}
\sum_{i\in I} u^i(q^i;(e^i-q^i)\cdot p) &= \sum_{i\in I} v^i(q^i) + \sum_{i\in I}\left(e^i - q^i\right)\cdot p \\
&= \sum_{i\in I} v^i(q^i) + \left(\sum_{i\in I} e^i - \sum_{i\in I} q^i\right)\cdot p \\
&= \sum_{i\in I} v^i(q^i) \\
&\leq \sum_{i\in I} v^i(\hat{q}^i) + \varepsilon \\
&= \sum_{i\in I} v^i(\hat{q}^i) + \left(\sum_{i\in I} e^i - \sum_{i\in I} \hat{q}^i\right)\cdot p + \varepsilon \\
&= \sum_{i\in I} v^i(\hat{q}^i) + \sum_{i\in I}\left(e^i - \hat{q}^i\right)\cdot p + \varepsilon \\
&= \sum_{i\in I} u^i(\hat{q}^i;(e^i-\hat{q}^i)\cdot p) + \varepsilon. \qquad (6)
\end{aligned}$$

Grouping (6) by agents, we have

$$\sum_{i\in I}\left(u^i(q^i;(e^i-q^i)\cdot p) - u^i(\hat{q}^i;(e^i-\hat{q}^i)\cdot p)\right) \leq \varepsilon. \qquad (7)$$

As $[q;p]$ is a δ-equilibrium, we have

$$\sum_{i\in I}\left(\max_{\bar{q}^i\in\mathbb{Z}^\Gamma}\left\{u^i(\bar{q}^i;(e^i-\bar{q}^i)\cdot p)\right\}-u^i(q^i;(e^i-q^i)\cdot p)\right)\leq\delta. \tag{8}$$

Summing (7) and (8), we have

$$\sum_{i\in I}\left(\max_{\bar{q}^i\in\mathbb{Z}^\Gamma}\left\{u^i(\bar{q}^i;(e^i-\bar{q}^i)\cdot p)\right\}-u^i(\hat{q}^i;(e^i-\hat{q}^i)\cdot p)\right)\leq\delta+\varepsilon.$$

Hence, $[\hat{q};p]$ is a $(\delta+\varepsilon)$-equilibrium, as claimed. □

By Lemma 2, $[\hat{q};p]$ is a $(\delta+\varepsilon)$-equilibrium. As $\hat{q}^j=e^j$ for all $j\in J$ by assumption, we have

$$\sum_{j\in J}\left(\max_{\bar{q}^j\in\mathbb{Z}^\Gamma}\left\{u^j(\bar{q}^j;(e^j-\bar{q}^j)\cdot p)\right\}-u^j(e^j;0)\right)\leq\delta+\varepsilon. \tag{9}$$

Meanwhile, with $t^j=(e^j-q^j)\cdot p$, we have

$$u^j(q^j;t^j)=u^j(q^j;(e^i-q^i)\cdot p)\leq\max_{\bar{q}^j\in\mathbb{Z}^\Gamma}\left\{u^j(\bar{q}^j;(e^j-\bar{q}^j)\cdot p)\right\}. \tag{10}$$

Combining (9) and (10), we see that

$$\sum_{j\in J}\left(u^j(q^j;t^j)-u^j(e^j;0)\right)\leq\sum_{j\in J}\left(\max_{\bar{q}^j\in\mathbb{Z}^\Gamma}u^j(\bar{q}^j;(e^j-\bar{q}^j)\cdot p)-u^j(e^j;0)\right)\leq\delta+\varepsilon,$$

as desired.

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