On the Correspondence of Contracts to Salaries in (Many-to-Many) Matching

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Abstract

In this note, I extend the work of Echenique (forthcoming) to show that a model of many-to-many matching with contracts may be embedded into a model of many-to-many matching with wage bargaining whenever (1) all agents' preferences are substitutable and (2) the matching with contracts model is *unitary*, in the sense that every contractual relationship between a given firm–worker pair is specified in a single contract. Conversely, I show that unitarity is essentially necessary for the embedding result.

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Since its introduction, the matching with contracts model has been extensively generalized (Ostrovsky (2008); Hatfield and Kominers (2010, forthcoming)), and has been applied in surprising contexts such cadet–branch matching (Sönmez and Switzer (2011); Sönmez (2011)) and the Japanese medical residency match (Kamada and Kojima (2011, forthcoming)).[2] Echenique (forthcoming) has recently shown that under the substitutability condition crucial for many of the results of Hatfield and Milgrom (2005), the matching with contracts model directly embeds into the earlier, and seemingly less general, Kelso and Crawford (1982) model of many-to-one matching with salaries and gross substitutes preferences.

The key insight of Echenique (forthcoming) is that in many-to-one matching with contracts models, contract negotiations between a firm and worker are in a sense "separable" from other contract negotiations whenever that firm and worker have substitutable preferences. In this note, I observe that this insight—and hence the embedding

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[2]Hatfield and Milgrom (2005) introduced the matching with contracts model as a generalization of the Kelso and Crawford (1982) model of many-to-one matching with salaries; the possibility of such a generalization was first noted in remarks of Crawford and Knoer (1981) and Kelso and Crawford (1982).

result of Echenique (forthcoming)—applies more generally.

Specifically, I show that bargaining over contracts is isomorphic to a firm–worker salary bargain whenever agents' preferences are substitutable and *a firm and worker are allowed to sign at most one contract with each other*. This latter condition, which I call *unitarity*, may be a natural assumption for (many-to-many) firm–worker matching markets in which each worker can serve in at most one role at each firm.[3] Unitarity is automatic in the many-to-one matching settings considered by Hatfield and Milgrom (2005) and Echenique (forthcoming), as in those settings workers have unit demand for contracts; it is also assumed in the many-to-many model of Klaus and Walzl (2009).[4]

I demonstrate (by example) that unitarity is essentially necessary for the Echenique (forthcoming) embedding result. Nonunitary many-to-many matching models such as that of Hatfield and Kominers (2010) need not correspond to wage bargaining models, even if all agents have substitutable preferences over contracts.

1. Preliminaries

1.1. Basic Models

Extending the framework of Echenique (forthcoming), I introduce two models of generalized matching among a (finite) set F of firms and a (finite) set W of workers.

1.1.1. Matching with Contracts

A model of *(many-to-many) matching with contracts* is specified by a set $X \subseteq F \times W \times T$ of contracts, where T is (finite) a set of contractual terms, along with a (one-to-one) utility function $u_i : 2^X \to \mathbb{R}$ for each $i \in F \cup W$. For a contract $x \in X$, I denote by x_F and x_W the firm and worker associated to x, respectively. For $Y \subseteq X$, I denote by $Y_i \equiv \{x \in Y : i \in \{x_F, x_W\}\}$ the set of contracts in Y associated to $i \in F \cup W$. Although the utility function of each agent $i \in F \cup W$ is defined (for convenience) over the entire space 2^X, I require that it depend only on the contracts actually available to i, that is, that $u_i(Y) = u_i(Y_i)$ for each $Y \subseteq X$.[5]

The utility function of $i \in F \cup W$ determines a *preference relation* P_i over sets of contracts $Y \subseteq X_i$ and an associated *choice function*

$$C_i(Y) \equiv \max_{P_i}\{Z : Z \subseteq Y_i\}$$

defined over sets of contracts $Y \subseteq X$.[6] The preferences of $i \in F \cup W$ are *substitutable*

[3]The market designer must exercise care, however, in defining the "roles" that a worker may serve. As Hatfield and Kominers (2010) demonstrate, the most natural contracting model for a matching market may involve splitting jobs into multiple distinct positions (e.g., a night shift and a day shift).

[4]Hence, my result extends the Echenique (forthcoming) approach to the setting of Klaus and Walzl (2009).

[5]Note that unlike the many-to-one matching with contracts models considered by Hatfield and Milgrom (2005) and Echenique (forthcoming), my model does not require that the utility functions of workers $w \in W$ exhibit *unit-demand*, i.e. that they be choice-equivalent to utility functions on the restricted space $\{\{x\} : X_i \cup \{\emptyset\}$.

[6]Here, I use the notation $\max_{P_i}$ to indicate that the maximization is taken with respect to the preferences of agent i.

if for any $x, z \in X$ and $Y \subseteq X$,

$$x \notin C_i(Y \cup \{x\}) \quad \Longrightarrow \quad x \notin C_i(Y \cup \{x, z\}).^7$$

A set of contracts $Y \subseteq X$ is called an *allocation*. An allocation $Y \subseteq X$ is said to be *individually rational* if $C_i(Y) = Y_i$ for all $i \in F \cup W$, and is said to be *unblocked* if there does not exist a nonempty set of contracts $Z \nsubseteq Y$ such that $Z_i \subseteq C_i(Y \cup Z)$ for all $i \in F \cup W$. An allocation is *stable* if it is both individually rational and unblocked.[8]

1.1.2. Matching with Salaries

A model of *(many-to-many) matching with salaries* is specified by a set $S \subseteq \mathbb{R}_+$ of possible salaries, a (one-to-one) utility function $v_f : 2^{\{f\} \times W \times S} \to \mathbb{R}$ for each $f \in F$, and a (one-to-one) utility function $v_w : 2^{F \times \{w\} \times S} \to \mathbb{R}$ for each $w \in W$.[9] Without loss of generality, I restrict S to be finite, and identify it with the set of positive integers $\{1, \ldots, \bar{S}\}$ for some $\bar{S}$ suitably large that no worker is ever hired at salary $\bar{S}$, i.e. such that $v_f(B \cup \{(f, w, \bar{S})\}) < v_f(B)$ for all $B \subseteq (\{f\} \times (W \setminus \{w\}) \times S)$.

For each $f \in F$, the utility function v_f induces a demand function $D_f : S^{F \times W} \to 2^{\{f\} \times W \times S}$ defined by

$$D_f(s) \equiv \underset{B \subseteq \{f\} \times \{(w, s_{fw})\}}{\operatorname{argmax}} (v_f(B)).^{10}$$

The demand function $D_w : S^{F \times W} \to 2^{F \times \{w\} \times S}$ of each worker $w \in W$ is defined analogously. The demand function D_f of $f \in F$ satisfies the *gross substitutes* condition if, for any two salary matrices s and s' with $s_f \leq s'_f$,

$$(f, w, s_{fw}) \in D_f(s) \quad \Longrightarrow \quad (f, w, s'_{fw}) \in D_f(s')$$

for any $w \in W$ for which $s_{fw} = s'_{fw}$. Analogously, the demand function D_w of $w \in W$ satisfies the *gross substitutes* condition if, for any two salary matrices s and s', for which $s_w \geq s'_w$,

$$(f, w, s_{fw}) \in D_w(s) \quad \Longrightarrow \quad (f, w, s'_{fw}) \in D_w(s')$$

for any $f \in F$ for which $s_{fw} = s'_{fw}$.

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[7] Intuitively, this condition means that there are no two contracts $x, z \in X$ which are sometimes "complements" in the sense that the availability of z makes x more attractive.

[8] A number of alternative stability concepts are available for many-to-many matching settings (Blair (1988); Echenique and Oviedo (2006); Klaus and Walzl (2009)). In general, the choice of solution concept is somewhat immaterial for our exercise: embedding results analogous to Theorem 2 hold so long as the stability concept under consideration for the cases of matching with salaries corresponds to that considered for the case of matching with contracts. Nevertheless, I fix a choice of stability concept for concreteness, using that of Hatfield and Kominers (2010, forthcoming), which is neither weaker nor stronger than the *setwise stability* concept of Echenique and Oviedo (2006).

[9] I use the convention that utility functions for models of matching with contracts are denoted u, while those for models of matching with salaries are denoted v.

[10] Note that the demand of f only depends upon the salaries s_{fw} of workers w at f.

A *(salary) matching* is a set $B \subseteq F \times W \times S$ for which

$$(f, w, s_{fw}), (f, w, s'_{fw}) \in B \quad \implies \quad s_{fw} = s'_{fw},$$

i.e. a set of firm–worker pairs which specifies a unique salary for each pair. A model of matching with salaries may be viewed as a specialized model of matching with contracts having contract set $X = F \times W \times S$ and the choice constraint that

$$(f, w, s_{fw}), (f, w, s'_{fw}) \in C_i(B) \quad \implies \quad s_{fw} = s'_{fw}$$

(for all $i \in F \cup W$ and $B \subseteq F \times W \times S$):[11] Defining the function

$$\kappa_i(s) \equiv \begin{cases} \{(i, w, s_{iw}) : w \in W\} & i \in F \\ \{(f, i, s_{fi}) : f \in F\} & i \in W, \end{cases}$$

gives $D_i(s) = C_i(\kappa_i(s))$. With this observation, salary matchings naturally inherit the notions of *individual rationality*, *unblockedness*, and *stability* described above.[12]

1.2. Key Conditions for the Embedding Result

I introduce two conditions key to the embedding argument: unitarity and Pareto separability. To the best of my knowledge, the former of these two conditions has not been emphasized previously, although it has been discussed abstractly in the literature (Hatfield and Kominers (2010)). The latter condition was introduced by Hatfield and Kojima (2010).

Definition 1. A model of matching with contracts is *unitary* if for all $Y \subseteq X$,

$$\begin{aligned} \forall f \in F, \qquad x, x' \in C_f(Y) \quad &\implies \quad x_W \neq x'_W, \\ \forall w \in W, \qquad x, x' \in C_w(Y) \quad &\implies \quad x_F \neq x'_F. \end{aligned}$$

Intuitively, unitarity corresponds to the requirement that a firm and worker sign at most one contract with each other. All many-to-one matching models are unitary, as in such settings all workers w have *unit demand*: $|C_w(Y)| \leq 1$ for all $Y \subseteq X$.[13] In modeling many-to-many matching with contracts, unitarity is an added assumption; Klaus and Walzl (2009) impose it, while Hatfield and Kominers (2010) do not.

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[11] This identification is immaterial to my substantive results; I make it only to simplify the task of defining stability for many-to-many matching models with salaries.

[12] Thus, like in the definition of stability for matching models with contracts presented above, our stability concept for salary matchings allows agents in a deviating coalition to deviate while maintaining previous relationships with nondeviating agents. As remarked in Footnote 8: Alternative stability solution concepts are available, but the choice of stability concept is not crucial for the existence of an analog of the Echenique (forthcoming) embedding result.

[13] The unit demand condition immediately implies the second condition of unitarity. Additionally, it implies that there is no firm–worker pair $(f, w) \in F \times W$ such that $|X_w \cap C_f(Y)| > 1$, so the first condition of unitarity holds, as well.

Hatfield and Kojima (2010) show (in their Theorem 3) that in unitary models of matching with contracts, substitutability implies the following Pareto separability condition similar to that of Roth (1984).

Definition 2. The preferences of agent $i \in F \cup W$ are *Pareto separable* if for any $x, x' \in X_i$ with $x \neq x'$, $x_F = x'_F$, and $x_W = x'_W$,

$$\begin{aligned} &\exists Y \subseteq X \text{ such that } x \in C_i(Y \cup \{x, x'\}) \\ \implies \quad &\nexists Y' \subseteq X \text{ such that } x' \in C_i(Y' \cup \{x, x'\}). \end{aligned} \tag{1}$$

An equivalent formulation of Pareto separability is a form of "marginal-utility monotonicity" of contracts: the preferences of $i \in F \cup W$ are Pareto separable if and only if for any $x, x' \in X_i$ with $x \neq x'$, $x_F = x'_F$, and $x_W = x'_W$,

$$\begin{aligned} &\exists Y \subseteq X \text{ such that } u_i(Y \cup \{x\}) > u_i(Y \cup \{x'\}) \\ \implies \quad &\nexists Y' \subseteq X \text{ such that } u_i(Y' \cup \{x\}) < u_i(Y' \cup \{x'\}). \end{aligned} \tag{2}$$

It is clear that (1) holds if and only if (2) does. This latter formulation of Pareto separability, (2), leads to an embedding generalizing that of Echenique (forthcoming) whenever agents' preferences are subsitutable.

2. Generalizing the Echenique Embedding

For a model $(X, (u_i))$ of matching with contracts and a model $(S, (v_i))$ of matching with salaries, an *embedding* of $(X, (u_i))$ into $(S, (v_i))$ is a one-to-one function

$$g : X \hookrightarrow F \times W \times S$$

with $g(x) \in \{x_F\} \times \{x_W\} \times S$ for each $x \in X$. For $Y \subseteq X$ and an embedding g of $(X, (u_i))$ into $(S, (v_i))$, I say that $g(Y)$ *defines a (stable) matching in* $(S, (v_i))$ if $(f, w, s_{fw}), (f, w, s'_{fw}) \in g(Y)$ implies that $s_{fw} = s'_{fw}$ (and $g(Y)$ is stable in $(S, (v_i))$).

Theorem 1. *Suppose that $(X, (u_i))$ is a model of matching with contracts in which the preferences of each agent $i \in F \cup W$ are substitutable and Pareto separable. Then, there is a model of matching with salaries $(S, (v_i))$ and an embedding g of $(X, (u_i))$ into $(S, (v_i))$ such that*

1. *for each $i \in F \cup W$, the demand function D_i defined by v_i satisfies the gross substitutes condition, and*
2. *$Y \subseteq X$ is a stable allocation in $(X, (u_i))$ if and only if $g(Y)$ defines a stable matching in $(S, (v_i))$.*

The proof of Theorem 1 closely follows the argument of Echenique (forthcoming); I present it in Appendix A.

As substitutability implies Pareto separability in unitary matching with contracts models (Theorem 3 of Hatfield and Kojima (2010)), the following result is an immediate corollary of Theorem 1.

Theorem 2. *Suppose that* $(X, (u_i))$ *is a* unitary *model of matching with contracts in which the preferences of each agent* $i \in F \cup W$ *are substitutable. Then, the conclusions of Theorem 1 hold.*

Theorem 2 generalizes the embedding theorem of Echenique (forthcoming) to unitary many-to-many matching models such as that of Klaus and Walzl (2009). One consequence of Theorem 2 is the existence of stable allocations in unitary many-to-many matching with contracts models with substitutable preferences; this follows from existing results on the existence of stable outcomes in many-to-many salary matching models with substitutable preferences.[14]

3. The Importance of Unitarity

Pareto separability is key to the proof of Theorem 1; it is apparently the only preference condition needed (in addition to substitutability) for the embedding argument to work. Conversely, Pareto separability appears to be essential for the embedding result.

As the Pareto separability of substitutable preferences generally breaks down in nonunitary matching models, it is not clear what generalization of Theorem 2, if any, can be found for such models. Indeed, the embedding result fails even in simple nonunitary many-to-many matching with contracts settings. For example,[15] suppose that $F = \{f\}$, $W = \{w_1, w_2\}$, $X = \{x^{1,m}, x^{1,n}, x^{2,m}, x^{2,n}\}$, and that the preferences of f take the form

$$P_f : \{x^{1,m}, x^{1,n}\} \succ \{x^{1,m}, x^{2,n}\} \succ \{x^{2,m}, x^{1,n}\} \succ \{x^{2,m}, x^{2,n}\} \succ \{x^{1,m}\} \succ \{x^{1,n}\} \succ \{x^{2,m}\} \succ \{x^{2,n}\} \succ \emptyset.$$

In this example, the preferences of f are substitutable but are not Pareto separable[16] and (whenever w_1 and w_2 find all their contracts acceptable) even this simple nonunitary example cannot be embedded into a model of matching with salaries in such a fashion that the relevant substitutability and stability notions correspond.

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[14]Hatfield et al. (2011) show that stable outcomes exist (in the presence of substitutable preferences) in a model which includes many-to-many salary matching as a special case.

[15]The example I present has a natural interpretation: There are two workers, w_1 and w_2, the first of whom is more talented than the second. The firm has two jobs to fill, a morning position (m) a night position (n), and can hire at most one worker for each job. The firm prefers to have the best possible worker in the morning shift. It would like to fill both jobs if possible, but failing that would prefer to hire w_1 if at all possible.

[16]Taking $i = f$, $Y = \{x^{1,m}\}$, $Y' = \{x^{1,n}\}$, $x = x^{2,n}$, and $x' = x^{2,m}$ gives a violation of condition (1). This example also shows that in nonunitary models not even *sets* of contracts can be marginal-utility ranked in a fashion analogous to condition (2).

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Appendix A. Proof of Theorem 1

For $(f, w) \in F \times W$, I denote by $\check{X}_{fw}$ the set of *undominated* contracts in $X_f \cap X_w$, i.e. the set defined by

$$\check{X}_{fw} \equiv \{x \in X_f \cap X_w : \nexists x' \in X_f \cap X_w \text{ such that } u_f(\{x'\}) > u_f(\{x\}) \text{ and } u_w(\{x'\}) > u_w(\{x\})\}.$$

I order the contracts $x^1_{fw}, \ldots, x^{|\check{X}_{fw}|}_{fw} \in \check{X}_{fw}$ such that

$$\ell > \ell' \quad \Longrightarrow \quad u_w(\{x^\ell_{fw}\}) > u_w(\{x^{\ell'}_{fw}\}). \tag{A.1}$$

Note that (A.1) implies that

$$\ell > \ell' \quad \Longrightarrow \quad u_f(\{x^\ell_{fw}\}) < u_f(\{x^{\ell'}_{fw}\}), \tag{A.2}$$

since otherwise $x^{\ell'}_{fw}$ is dominated.

I let $\bar{S} = 1 + \max_{f \in F, w \in W} |\check{X}_{fw}|$, set $S = \{1, \ldots, \bar{S}\}$, and let $g : X \to F \times W \times S$ be the map with takes the contract $x^\ell_{fw} \in \check{X}_{fw}$ to the triple $(f, w, \ell) \in F \times W \times S$, and takes all dominated contracts x to $(x_F, x_W, \bar{S})$.[17]

I set $v_i(g(C_i(Y))) = u_i(C_i(Y))$ for each $Y \subseteq X$,[18] and set each v_i to equal $-1 + \min_{Y \subseteq X} u_i(Y)$ everywhere else in its domain. By construction, $(S, (v_i))$ is a model of matching with salaries, and g is an embedding of $(X, (u_i))$ into $(S, (v_i))$ such that $Y \subseteq X$ is stable in $(X, (u_i))$ if and only if $g(Y)$ defines a stable matching in $(S, (v_i))$.

To complete the proof, I now show that for each $i \in F \cup W$, the demand function D_i defined by v_i satisfies the gross substitutes condition. I show that D_f satisfies the gross substitutes condition for each $f \in F$; the proof that each D_w ($w \in W$) satisfies the gross substitutes condition is completely analogous.

For a salary vector $s_f \in S^W$, I let $X^{s_f}_f \equiv \{x^{s_{fw}}_{fw} : w \in W\}$ be the set of contracts in $X_f \cap (\cup_{w \in W} X_w)$ associated with the salaries s_{fw}, using the convention that if $|\check{X}_{fw}| < s_{fw}$ then $x^{s_{fw}}_{fw}$ is some arbitrary dominated contract in $X_f \cap X_w$.[19] Note that $D_f(s) = g(C_f(X^{s_f}_f))$.

Now, let

$$X^{s_f+}_f \equiv \left\{ x^{s'_{fw}}_{fw} : \bar{S} \geq s'_{fw} \geq s_{fw} \right\}.$$

As the preferences of f are Pareto separable, this immediately implies that $C_f(X^{s_f+}_f) \subseteq C_f(X^{s_f}_f)$.[20] It follows that $C_f(X^{s_f+}_f) = C_f(X^{s_f}_f)$, as $X^{s_f+}_f \supseteq X^{s_f}_f$.

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[17] Note that this map g is completely analogous to the embedding of Echenique (forthcoming). As in that embedding, for g to be one-to-one I must, strictly speaking, expand S and choose distinct assignments $(x_F, x_W, \bar{S} + \ell)$ for each dominated $x \in X$; I suppress this concern for notational convenience.

[18] This is well-defined because the matching with contracts model is unitary.

[19] Since dominated contracts are never chosen, it is without loss of generality to assume that $(X_f \cap X_w) \setminus \check{X}_{fw}$ is always nonempty.

[20] To see this, suppose otherwise—that there is some $x^{s^+_{fw}} \in C_f(X^{s_f+}_f) \setminus C_f(X^{s_f}_f)$. Then, there must

Now, let $s_f \leq s'_f$ be salary vectors for which $s_{fw} = s'_{fw}$ and $(f, w, s_{fw}) \in D_f(s)$. I must show that $(f, w, s'_{fw}) \in D_f(s')$. Suppose otherwise for the sake of contradiction, and note that

$$x_{fw}^{s'_{fw}} \notin C_f\left(X_f^{s'_f}\right) = C_f\left(X_f^{s'_f+}\right). \tag{A.3}$$

Since $X_f^{s'_f+} \subseteq X_f^{s_f+}$ and the preferences of f are substitutable, (A.3) implies that

$$x_{fw}^{s'_{fw}} \notin C_f(X_f^{s_f+}) = C_f(X_f^{s_f}).$$

But since $s'_{fw} = s_{fw}$, this means that $x_{fw}^{s_{fw}} \notin C_f(X_f^{s_f})$, contradicting the fact that

$$(f, w, s_{fw}) \in D_f(s) = g(C_f(X_f^{s_f})).$$

be some $x^{s_{fw}} \in C_f(X_f^{s_f})$ with $w = x_W^{s_{fw}^+} = x_W^{s_{fw}}$; Pareto separability (in the form (2)) then shows that

$$u_f\left(\left\{x^{s_{fw}^+}\right\}\right) > u_f\left(\{x^{s_{fw}}\}\right).$$

But then, by (A.2), $s_{fw}^+ < s_{fw}$, contradicting the fact that $x^{s_{fw}^+} \in X_f^{s_f+}$.