Optimizing Reserves in School Choice: A Dynamic Programming Approach

Franklyn Wang[a,*], Ravi Jagadeesan[b], Scott Duke Kominers[b,c]

[a] Harvard College
[b] Harvard Business School and Department of Economics, Harvard University
[c] Center of Mathematical Sciences and Applications and Center for Research on Computation and Society, Harvard University, and National Bureau of Economic Research

Abstract

We introduce a new model of school choice with reserves in which a social planner is constrained by a limited supply of reserve seats and tries to find an optimal matching according to a social welfare function. We construct the optimal distribution of reserves via a quartic-time dynamic programming algorithm. Due to the modular nature of the dynamic program, the mechanism is *strategy-proof* for reserve-eligible students.

Keywords: Matching, Reserves, Dynamic Programming
JEL: C61, C78

1. Introduction

College admissions and school choice are often modeled as two-sided stable matching problems, where students have preferences over schools and schools have priorities over students.

Social planners sometimes wish to optimize the school choice matchings in line with overarching social welfare objectives. Since welfare objectives are generally monotonic in students' preferences for their assignments, the deferred acceptance algorithm produces the unique stable matching which maximizes any welfare function for fixed student preferences and priority rules over stable matchings [1]. However, if social planners are allowed to adjust the priorities at each seat, they can produce matchings with higher social welfare than the matching from deferred acceptance. One typical modification is to designate some seats as *reserve*, at which a targeted group of students has higher priority.

To avoid completely distorting away from priority-based assignment, usually some limit on the number of reserve seats is respected. Prior work has achieved this by distributing reserves to individual schools upfront—making a fixed number of seats at each school reserved for the targeted group [2, 3, 4, 5]. However, in some situations we need only respect a total limit on the number of reserve seats. We could in principle adjust reserves across schools

*Corresponding author
Email addresses: franklyn_wang@college.harvard.edu (Franklyn Wang), ravi.jagadeesan@gmail.com (Ravi Jagadeesan), kominers@fas.harvard.edu (Scott Duke Kominers)

flexibly if that would improve welfare; this would ensure that reserve seats are not wasted at schools that targeted students do not wish to attend.

This paper designs a mechanism that distributes reserves across schools to maximize social welfare. We consider a model in which students are ordered according to test scores. There are two types of students: *targeted* students, who are eligible for reserve seats, and *non-targeted* students, who are not. At reserve seats, all targeted students receive a certain number of extra points. The social planner's objective is to first distribute a bounded number of reserves across schools and then assign students to schools so as to maximize a lexicographic welfare function, whereby the social planner wishes to maximize the placements of top-ranking targeted students. Crucially, unlike in prior work [6, 7, 8, 9], the distribution of reserves to schools is endogenously determined alongside the assignment of students to schools.

We use a polynomial-time dynamic programming approach to determine the optimal distribution of reserves and the student-optimal stable assignment under those reserves. Our mechanism, which we call *dynamically programmed reserve allocation (DPRA)*, is strategy-proof for targeted students but not strategy-proof for non-targeted students (even in the case where test score boosts for reserve-eligible students are infinite); since there is a unique welfare-maximizing stable matching under our lexicographic welfare model, this implies that no stable and strategy-proof mechanism can maximize welfare.

Our analysis may help social planners balance the needs of targeted and non-targeted students. The DPRA mechanism places reserves where targeted students need them, but ensures that non-targeted students are not harmed too much by respecting a limit on the total number of reserves. The strategy-proofness of DPRA for targeted students is normatively appealing, as it removes strategic burden from the students that the social planner is trying to help [10, 11].

From a practical perspective, our model of reserves captures a feature of the Chinese college admissions system whereby certain students (like athletes, prodigies, and members of minority groups) receive elevated priority at certain seats [12]. The elevated priority is given in the form of a boost in score on the *gaokao*—the Chinese college entrance examination—that is applicable at seats called *independent enrollment seats*. Currently, the distribution of independent enrollment seats is fixed; our results suggest how the distribution can be endogenously optimized within the student–school assignment process to optimize welfare.

The remainder of this paper is organized as follows. Section 2 gives an illustrative example. Section 3 describes the model formally. Section 4 describes our mechanism. Section 5 concludes. Appendix A presents the proofs.

2. An Illustrative Example

In this section, we provide an example that illustrates how the distribution of reserve seats across schools can affect the outcome of deferred acceptance. We also describe the optimal reserve distributions.

Suppose that there are four students—s_1, s_2, s_3, and s_4—and four schools—c_1, c_2, c_3, and c_4. Each school has a capacity to enroll one student. Students s_1 and s_2 are targeted by the reserve policy and hence receive priority over students s_3 and s_4 at all reserve seats.

If not allocated a reserve, all schools rank students according to the master priority ordering

$$s_3 \succ_{c_i} s_4 \succ_{c_i} s_1 \succ_{c_i} s_2.$$

When a school is allocated a reserve, its priority ordering becomes

$$s_1 \succ_{c_i} s_2 \succ_{c_i} s_3 \succ_{c_i} s_4$$

—ranking the targeted students above the non-targeted students. The students' preferences are given by

$$\begin{aligned} s_1 &: c_1 \succ_{s_1} c_2 \succ_{s_1} c_3 \succ_{s_1} c_4 \\ s_2 &: c_2 \succ_{s_2} c_1 \succ_{s_2} c_4 \succ_{s_2} c_3 \\ s_3 &: c_2 \succ_{s_3} c_1 \succ_{s_3} c_3 \succ_{s_3} c_4 \\ s_4 &: c_1 \succ_{s_4} c_2 \succ_{s_4} c_4 \succ_{s_4} c_3 \end{aligned}$$

We now illustrate some of the subtleties of how the allocation of reserves to schools can affect the outcome of deferred acceptance.

- Scenario 1: No reserve is allocated. In this case, in the outcome of deferred acceptance, student s_1 is matched to c_3, student s_2 is matched to c_4, student s_3 is matched to c_2 and student s_4 is matched to c_1.

- Scenario 2: School c_1 is allocated a reserve. In the case, in the outcome of deferred acceptance, student s_1 is matched to c_1, student s_2 is matched to c_3, student s_3 is matched to c_2, and student s_4 is matched to c_4.

  Compared with Scenario 1, allocating a reserve to c_1 helps s_1 to obtain a seat at c_1 but causes s_2 to lose her seat at s_4. Intuitively, allocating a reserve to c_1 causes s_4 to lose her seat at c_1, therefore causing her to seek a seat at c_4. Because there is no reserve at c_4, student s_4 has higher priority than s_2 at c_4.

- Scenario 3: School c_2 is allocated a reserve. In the case, in the outcome of deferred acceptance, student s_1 is matched to c_2, student s_2 is matched to c_3, student s_3 is matched to c_1, and student s_4 is matched to c_4.

  Compared with Scenario 1, s_1 obtains a more desirable assignment and student s_2 obtains a less desirable assignment. Compared with Scenario 2, all students are hurt. Intuitively, moving the reserve from c_1 to c_2 hurts all students because s_1 not getting his first choice causes increased competition at all seats.

- Scenario 4: Schools c_1 and c_2 are allocated reserves. In this case, in the outcome of deferred acceptance, student s_1 is matched to c_1, student s_2 is matched to c_2, student s_3 is matched to c_3, and student s_4 is matched to c_4.

  This allocation is weakly better than all of the above allocations for both s_1 and s_2, since s_1 and s_2 both are protected by the reserve seats at c_2 and c_1.

Consider a social welfare objective that values placements of targeted students lexicographically more than the placements of non-targeted students. With one reserve, it is optimal to allocate the reserve to c_1, as in Scenario 2. With two reserves, it is optimal to allocate the reserves to c_1 and c_2, as in Scenario 4. It is never optimal to allocate a single reserve to school c_2, as in Scenario 3, because all students would prefer for the reserve to be moved to c_1.

For a given number of available reserve seats, our algorithm endogenously determines how best to allocate them—choosing Scenario 2 in the case of one reserve, and Scenario 4 in the case of two.

3. Model

3.1. Matching with Reserves

There is a set $S = \{s_1, s_2, \ldots, s_n\}$ of students and a set $C = \{c_1, c_2, \ldots, c_m\}$ of schools. For sake of simplicity, we assume that each school has unit capacity. Reserve-eligible students are identified via a characteristic function $\mathcal{T} : S \to \{0, 1\}$: a student s is *reserve-eligible* if $\mathcal{T}(s) = 1$.

Each student s_i has a *test score* $\tau(s_i) \in \mathbb{R}$. Let $\sqsupset^x$ be the priority order where students are ranked by test scores, with each reserve-eligible student receiving x bonus points. If $\tau(s) + x\mathcal{T}(s) > \tau(s') + x\mathcal{T}(s)$, then $s \sqsupset^x s'$. The priority ordering of school c is

$$\succ_c = \begin{cases} \sqsupset^0 \text{ if there is no reserve at } c \\ \sqsupset^{x_c} \text{ if there is a reserve at } c \end{cases}, \tag{1}$$

where $x_c \geq 0$ is the number of bonus points given to reserve-eligible students at reserve seats from school c. We assume that test scores and the numbers of bonus points are such that each school's priority order is strict irrespective of whether the school is allocated a reserve.

Each student has a strict preference ordering over C. If student s prefers school c to school c', then we write $c \succ_s c'$. Define the *preference profile* as

$$(\succ_{s_i})_{s_i \in S} = \succ_S .$$

Define the *priority profile* as

$$(\succ_{c_i})_{c_i \in C} = \succ_C .$$

A *matching* M is a set of matched pairs of students and schools in which every student and every school appears at most once. Let $M[s] = c$ if $(s, c) \in M$ and $M[s] = \varnothing$ if no such c exists. Also, let $M[c] = s$ if $(s, c) \in M$ and $M[c] = \varnothing$ if no such c exists. Furthermore, define $M[S']$ for a set of students S' as the set of schools that students in S' attend under M. Define $M[C']$ for a set of schools C' as the set of student that attend schools in C' under M. Formally, we define $M[S']$ and $M[C']$ as

$$M[S'] = \bigcup_{s \in S'} \{M[s]\} \text{ and } M[C'] = \bigcup_{c \in C'} \{M[c]\}.$$

An *outcome* is a tuple $(M, L, R, \succ_{S'}, C')$ consisting of a matching M; the preferences of the students in the outcome $\succ_{S'}$, where $S' \subseteq S$ is a set of students involved in the outcome; a set $C' \subseteq C$ of schools involved in the outcome; the set $L \subseteq C$ of locations of the reserves;

and the set $R \subseteq S$ of reserve-eligible students. A *blocking pair* is a student $s \in S'$ and a school $c \in C'$ such that $s \succ_c M[c]$ and $c \succ_s M[s]$, where $\succ_c$ was defined in (1)—i.e., a student and a school who each prefer each other to their assignment. An outcome is *stable* if it has no blocking pairs, and *good* if $L \subseteq M[R]$, meaning that reserve seats are only given to reserve-eligible students.

A *partial outcome* is a tuple $(M, R, \succ_{S'}, C')$ that includes all the components of an outcome except for the location of the reserve seats. For a partial outcome O, let O_M denote the matching, let O_R the set of reserve-eligible students, $O_{\succ_{S'}}$ the preference profile of the involved students, and $O_{C'}$ the set of involved schools. For a set of schools $L_0 \subseteq C$, we define $L_0 \ltimes O$ to be an outcome where the reserve seats are given by L_0, but the other parameters are all the same as O.

A partial outcome O is *T-feasible* if there exists a set of schools L_0 with size at most T such that $L_0 \ltimes O$ is a stable outcome—that is, if it is possible to allocate at most T reserves to make O stable. For a set of preferences $\succ_S$, a set of schools C, and a set of reserve-eligible students R, the set of matchings corresponding to T-feasible partial outcomes is denoted by $\mathcal{M}_F(\succ_S, C, R)$.

3.2. The Social Planner's Problem

The social planner evaluates outcomes using a welfare function that aggregates students' preferences using a social welfare metric. We restrict our attention solely to one specific lexicographic welfare function, which requires defining a relative order $\blacktriangleright$ over students, called precedence, corresponding to how much the social planner "cares" about a student. Formally, the *precedence order* $\blacktriangleright$ is the ordering of students in which

- all targeted students precede all non-targeted students, and
- within each group, higher-scoring students precede lower-scoring ones.

Let $\blacktriangleright_i$ be the ith ranked element of $\blacktriangleright$, and denote $\text{Pre}_k = \{\blacktriangleright_1, \blacktriangleright_2, \ldots, \blacktriangleright_k\}$.

Definition 1. A *$\blacktriangleright$-lexicographic welfare function* $W_{\blacktriangleright,\succ_S}$ is a function from matchings to real numbers such that $W_{\blacktriangleright,\succ_S}(\mathcal{A}) > W_{\blacktriangleright,\succ_S}(\mathcal{B})$ for matchings $\mathcal{A}$ and $\mathcal{B}$ if and only if there exists an integer $k \geq 0$ such that $\mathcal{A}[\blacktriangleright_i] = \mathcal{B}[\blacktriangleright_i]$ for all $i \leq k$ and $\mathcal{A}[\blacktriangleright_{k+1}] \succ_{\blacktriangleright_{k+1}} \mathcal{B}[\blacktriangleright_{k+1}]$, where preferences are given by $\succ_S$.

To maximize welfare under $W_\blacktriangleright$, we maximize $\blacktriangleright_1$'s result (according to her reported preferences), then $\blacktriangleright_2$'s result (according to her reported preferences), and so on. The social planner's objective is to maximize $W_{\blacktriangleright,\succ_S}(O_M)$ over all T-feasible partial outcomes. Note that this is purely a function of the matching, and not of the locations of the reserve seats or of reserve-eligible status. That is, the social planner has T reserves to distribute and seeks to maximize its social welfare metric among stable matchings.

3.3. Mechanisms

A *mechanism* is a function f that takes as input a preference profile, the set of schools, and the reserve eligible students and outputs a matching. A mechanism f is called *T-feasible* if $f(\succ_S, C, \mathcal{T})$ is T-feasible for all choices of parameters. A mechanism f is *strategy-proof for student s* if f makes truth-telling a dominant strategy for s—that is, if

$$f(\succ_S, C, \mathcal{T})[s] \succeq_s f((\succ_{S \setminus \{s\}}, \succ_s'), C, \mathcal{T})[s]$$

for all choices of parameters.

Definition 2. A T-feasible matching M is *socially optimal* if it has weakly higher welfare than any other T-feasible matching M', that is,

$$W_{\blacktriangleright, \succ_S}(M) \geq W_{\blacktriangleright, \succ_S}(M') \text{ for all } M' \in \mathcal{M}^G(\succ_S, C, \mathcal{T})$$

Definition 3. A T-feasible mechanism f is *socially optimal* if it returns socially optimal matchings.

Recall that among all matchings that are stable with respect to a given reserve distribution, there is a *student-optimal* matching that is weakly preferred by all students to any of the others [1]. Since $W_{\blacktriangleright}$ is monotonic in students' preferences, any socially optimal mechanism must return matchings that are student-optimal with respect to some reserve distribution (where the reserve distribution may depend endogenously on students' reported preferences).

The *deferred acceptance mechanism* [1] is an example of a mechanism. It proceeds as follows.

- At step 1, every student applies to his or her first choice school. Each school rejects all but its favorite student, who is temporarily held.
- At step $i > 1$, all students who have been rejected apply to their favorite school that has not yet rejected them. Again, each school rejects all but its favorite student, whom the school holds. (Note that it is possible for a school's temporarily held student to be rejected in a later round)
- When the students are all either matched to a school or have proposed to all schools they prefer to the outside option, the process ends.

We let $\mathcal{DA}(L, R, \succ_{S'}, C')$ denote the matching returned by deferred acceptance in the sub-market in which the set of students is S', the set of schools is C', reserves are set at the schools in L, the set of reserve-eligible students is R, and students' preferences are given by $\succ_{S'}$.

The deferred acceptance mechanism is strategy-proof, stable, and student-optimal for a fixed reserve distribution L [1, 13, 14]. Our contribution is to consider the case in which L is determined endogenously from students reports and set to maximize a welfare criterion.

4. The Dynamic Programming Reserve-Allocation Mechanism (DPRA)

In this section, we present the dynamic programming reserve-allocation mechanism, which solves the social planner's problem. The general idea of the dynamic programming method is to start with a market with no reserve-eligible students and successively make targeted students reserve-eligible (in priority order), adjusting the matching as we do so.

The intuition behind the algorithm is that making a targeted student reserve-eligible will not affect the assignment of a targeted student of higher priority under $\sqsupset^0$, since the "chain of elimination" formed by the promotion of the former student will not affect the latter student. This mechanism finds the welfare-maximizing T-feasible matching, where the welfare function is $W_{\blacktriangleright,\succ_S}$ and the precedence order $\blacktriangleright = \sqsupset^\infty$. Our algorithm runs in polynomial time.

If opt$[i]$ is the welfare-maximizing T-feasible matching when just the students in Pre_i are reserve-eligible, then we seek to find opt$[b]$, where b is the number of targeted students. Formally, finding opt$[i]$ is just calculating

$$\text{opt}[i] = \underset{\mathcal{M}_F(\succ_S, C, \text{Pre}_i)}{\text{argmax}} W_{\blacktriangleright,\succ_S}$$

We do this by finding each of opt$[i]$ where $0 \leq i \leq b$ sequentially. Transitioning from opt$[i]$ to opt$[i+1]$ corresponds to making targeted student $\blacktriangleright_{i+1}$ reserve-eligible.

4.1. Structural Observations

We make the following observation about the opt$[i]$, which shows that making student $\blacktriangleright_{i+1}$ reserve-eligible does not affect the assignments of students in Pre_i.

Proposition 1. We have

$$\text{opt}[i][\blacktriangleright_j] = \text{opt}[i+1][\blacktriangleright_j]$$

for all $j \leq i$.

Proposition 1 is proven by noting that the vacancy chain created by $\blacktriangleright_{i+1}$'s promotion cannot influence $\blacktriangleright_1, \blacktriangleright_2, \ldots, \blacktriangleright_i$. Proposition 1 means that if we have found opt$[i]$, it is substantially easier to determine opt$[i+1]$. To solve the reserve-distribution problem, we just need to determine what the assignment of $\blacktriangleright_{i+1}$ is when transitioning between opt$[i]$ and opt$[i+1]$, which we do by checking by iterating through $\blacktriangleright_{i+1}$'s preference list in descending order, and then checking whether that matching can be stabilized with at most T reserve seats; in the next section, we introduce a property that formalizes this idea.

4.2. The Minimum Feasible Number

Let the *minimum feasible number* of a partial outcome $(M, R', \succ_{S'}, C')$ where $R' \subseteq S'$ be the smallest integer k such that there exists a matching M' such that $(M', R', \succ_S, C)$ is k-feasible and $M[s] = M'[s]$ for all $s \in S'$ and $M[c] = M'[c]$ for all $c \in C'$. If no such integer exists, let the minimum feasible number be ∞.

4.3. The Main Algorithm

Let there be a non-targeted students, b targeted students, and T total reserve slots over m schools. We use dynamic programming as follows: Again, let opt$[i]$ the lexicographically optimal T-feasible matching in which exactly the students in Pre_i are reserve-eligible (there are still at most T reserve seats). As there are b targeted students in the original problem, the goal is to compute opt$[b]$. Our algorithm runs as follows. We will compute matchings dp$[0]$, dp$[1], \ldots,$ dp$[b]$. We will then show that dp$[i]$ = opt$[i]$, by proving that dp$[b]$ maximizes $W_{\blacktriangleright,\succ_S}$.

- Step 0: Let dp[0] be the result of deferred acceptance in which we ignore the targeted status of any student.
- Step i for $1 \leq i \leq b$: Consider the matching

$$M' = \bigcup_{j=1}^{i-1} \{(\blacktriangleright_j, \mathrm{dp}[i-1][\blacktriangleright_j])\}.$$

which represents the schools to which Pre_{i-1} are matched to under $\mathrm{dp}[i-1]$.

Let c^* be $\blacktriangleright_i$'s most-preferred school such that:

  - $\mathrm{dp}[i-1][\blacktriangleright_j] \succ_{\blacktriangleright_j} c^*$ for all $1 \leq j \leq i-1$, and
  - The minimum feasible number of $(M' \cup \{(\blacktriangleright_i, c^*)\}, \mathrm{Pre}_i, \succ_{\mathrm{Pre}_i}, \{c^*\} \cup M'[\mathrm{Pre}_{i-1}])$ is at most T.

c^* is the most preferred school that $\blacktriangleright_i$ can be matched to while keeping the minimum feasible number at most T.

Let $M_i^{c^*} = M' \cup \{(\blacktriangleright_i, c^*)\}$. Run deferred acceptance in the market consisting of $S \setminus \mathrm{Pre}_i$ and unmatched schools in C to obtain a matching M''. We set $\mathrm{dp}[i] = M_i^{c^*} \cup M''$.

We call the preceding algorithm *Dynamic Programming with Reserve Allocation (DPRA).*

Proposition 2. In the DPRA algorithm, $\mathrm{dp}[i] = \mathrm{opt}[i]$, that is, $\mathrm{dp}[i]$ optimizes welfare when only the students in Pre_i are reserve-eligible.

Corollary 1. Any T-feasible mechanism that optimizes welfare is outcome-equivalent to DPRA.

The proof of Proposition 2 is via an inductive argument on i, because Proposition 1 means that we can reuse $\mathrm{opt}[i]$ when computing $\mathrm{opt}[i+1]$.

We now show that DPRA is strategy-proof for targeted students; this relieves targeted students of the burden of having to strategize.

Proposition 3. The DPRA mechanism is strategy-proof for targeted students.

DPRA's strategy-proofness arises from the fact that it essentially only queries for $\blacktriangleright_i$'s preferences when computing the matching $\mathrm{dp}[i]$.

Proposition 4. The DPRA mechanism is not strategy proof for non-targeted students, even when $x_c \equiv \infty$.

Even though the mechanism is not strategy-proof for non-targeted students, the strategy-proofness for targeted students is normatively appealing, because it means that using DPRA makes the assignment process strategically simple for the students who the policymakers are trying to help. Moreover, there is no way to maximize welfare while requiring T-feasibility and strategy-proofness for all students.

Combining Corollary 1 and Proposition 4, we have the following results.

Corollary 2. No T-feasible and strategy-proof mechanism can maximize $W_{\blacktriangleright, \succ_S}$ for all preference profiles $\succ_S$.

Note that DPRA chooses the welfare-maximizing matching over all T-feasible matchings. As one T-feasible matching is the result of deferred acceptance with no reserves, DPRA will always yield a matching with weakly higher welfare.

4.4. Calculating the Minimum Feasible Number

The above analysis does not explain how to compute the minimum feasible number effectively. Here, we explain an algorithm which computes the minimum feasible number in polynomial time.

Consider the following algorithm, denoted the *Blocking Pairs Algorithm* (BPA), that is polynomial time and computes the minimum feasible number. BPA takes as input a partial outcome $O = (M, R, \succ_{S'}, C')$ where $R \subseteq S'$ and calculates the minimum feasible number. Consider the matching

$$M' = M \cup \mathcal{DA}(\varnothing, \varnothing, \succ_{S \setminus S'}, C \setminus C').$$

Call this the *auxiliary matching*. Now, if the outcome $(M', M'[R], R, \succ_S, C)$ is stable, BPA returns the number of schools in $M'[R]$ that are blocked by students in $S \setminus R$ in the outcome $(M', \varnothing, R, \succ_S, C)$. Otherwise, BPA returns ∞.

Note that BPA runs in $\mathcal{O}(mn)$ time. Intuitively, if there exists a set of reserves that will stabilize the matching, and we place reserve seats at the schools that are blocked in C', the matching will become stable.

It now suffices to show that BPA computes the minimum feasible number correctly.

Proposition 5. BPA computes the minimum feasible number of every partial outcome $(M, R, \succ_{S'}, C')$ with $R \subseteq S'$.

Proposition 6. The DPRA algorithm has polynomial runtime $\mathcal{O}(bm^2n)$.

5. Conclusion

This paper considered how a social planner can distribute reserves among schools to maximize a lexicographic social welfare objective $W_{\blacktriangleright, \succ_S}$. When the social welfare objective is lexicographic in targeted students' preferences, we provide a mechanism based on dynamic programming to distribute reserves and find a welfare-maximizing stable matching. Our mechanism is strategy-proof for targeted students, ensuring that optimizing the distribution of reserves does not create complex strategic incentives for targeted students.

It would be interesting to analyze more general social welfare objectives to better understand how schools schools systems should distribute reserves. It would also be interesting to consider endogenous reserve distribution in matching with contracts settings, for example to incorporate financial aid packages into our model.

Acknowledgments

This research was conducted while Wang was a Research Science Institute (RSI) Summer Scholar at the Massachusetts Institute of Technology, sponsored by the Center for Excellence in Education (CEE), and while Wang and Jagadeesan were appointed as Economic Design Fellows at the Harvard Center of Mathematical Sciences and Applications (CMSA). The authors thank Ava Chen, Maggie Chen, Shengwu Li, Dhruv Rohatgi, Alexander Teytelboym, Karthik Yegnesh, Parag Pathak, Daniel Michael, and especially Jenny Sendova for helpful comments and suggestions. Wang gratefully acknowledges Louise Epstein, Jeff Cui, and many others who have supported the CEE. Jagadeesan gratefully acknowledges the support of a National Science Foundation Graduate Research Fellowship. Kominers gratefully acknowledges the support of National Science Foundation grant SES-1459912 and the Ng Fund and the Mathematics in Economics Research Fund of the CMSA.

References

[1] D. Gale, L. S. Shapley, College admissions and the stability of marriage, American Mathematical Monthly 69 (1) (1962) 9–15.

[2] A. Abdulkadiroğlu, College admissions with affirmative action, International Journal of Game Theory 33 (4) (2005) 535–549.

[3] A. Abdulkadiroğlu, T. Sönmez, School choice: A mechanism design approach, American Economic Review 93 (3) (2003) 729–747.

[4] U. Dur, S. D. Kominers, P. A. Pathak, T. Sönmez, The demise of walk zones in Boston: Priorities vs. precedence in school choice, Journal of Political Economy 126 (6) (2018) 2457–2479.

[5] I. E. Hafalir, M. B. Yenmez, M. A. Yildirim, Effective affirmative action in school choice, Theoretical Economics 8 (2) (2013) 325–363.

[6] O. Aygun, B. Turhan, Dynamic reserves in matching markets: Theory and applications, working paper (2016).

[7] O. Aygün, B. Turhan, Large-scale affirmative action in school choice: Admissions to IITs in India, American Economic Review 107 (5) (2017) 210–213.

[8] Y. Kamada, F. Kojima, Efficient matching under distributional constraints: Theory and applications, American Economic Review 105 (1) (2014) 67–99.

[9] Y. Kamada, F. Kojima, Stability concepts in matching under distributional constraints, Journal of Economic Theory 168 (2017) 107–142.

[10] A. Abdulkadiroğlu, P. A. Pathak, A. E. Roth, T. Sönmez, Changing the Boston school choice mechanism: Strategy-proofness as equal access, working paper (2006).

[11] P. A. Pathak, T. Sönmez, Leveling the playing field: Sincere and sophisticated players in the Boston mechanism, American Economic Review 98 (4) (2008) 1636–1652.

[12] S. Zou, Extra points for gaokao are reduced, China Daily.
URL `http://global.chinadaily.com.cn/a/201803/22/WS5ab2ff73a3105cdcf651376d.html`

[13] L. E. Dubins, D. A. Freedman, Machiavelli and the Gale-Shapley algorithm, American Mathematical Monthly 88 (7) (1981) 485–494.

[14] A. E. Roth, The economics of matching: Stability and incentives, Mathematics of Operations Research 7 (4) (1982) 617–628.

[15] M. Balinski, T. Sönmez, A tale of two mechanisms: Student placement, Journal of Economic Theory 84 (1) (1999) 73–94.

[16] V. P. Crawford, Comparative statics in matching markets, Journal of Economic Theory 54 (2) (1991) 389–400.

[17] T. Sönmez, M. U. Ünver, Matching, allocation, and exchange of discrete resources, in: Handbook of social Economics, Vol. 1, Elsevier, 2011, pp. 781–852.

Appendix A. Proofs

Appendix A.1. Preliminaries: Classes of Matchings

We say that a partial outcome $O = (M, R, \succ_{S'}, C')$ is *T-attainable* if there exists a reserve distribution L_0 with size at most T such that $\mathcal{DA}(L_0, R, \succ_{S'}, C') = M$. By construction, every T-attainable outcome is T-feasible. Due to the optimality of deferred acceptance among all stable mechanisms, every welfare-maximizing, T-feasible partial outcome must be T-attainable.

We say that a partial outcome $O = (M, R, S', C')$ is *good T-feasible* if there exists a reserve distribution L_0 with size at most T such that $L_0 \ltimes O$ is a stable and good outcome—that is, if it is possible to allocate at most T reserves to schools that enroll reserve-eligible students to obtain stability. Analogously, we say that a partial outcome O is *good T-attainable* if there exists a reserve distribution L_0 with size at most T such that $\mathcal{DA}(L_0, R, \succ_{S'}, \succ_{C'}) = M$, and furthermore the outcome $L_0 \ltimes O$ is stable and good.

Define the set of matchings present in at least one T-feasible (resp. T-attainable, good T-feasible, good T-attainable) partial outcome to be $\mathcal{M}_F(\succ_{S'}, C', R)$ (resp. $\mathcal{M}_A(\succ_{S'}, C', R)$, $\mathcal{M}_F^G(\succ_{S'}, C', R)$, $\mathcal{M}_A^G(\succ_{S'}, C', R)$), where the set of students is S', the set of schools is C', and the reserve-eligible students are exactly those in R.

Appendix A.2. Key Lemmata

Many of the proofs of the propositions depend on lemmata given here. The first lemma here allows us to interpret maximizing welfare as finding the optimal matching from many different sets; this allows us to have more flexibility when approaching other problems.

Lemma 1. *For all stable outcomes $O = (M, L, R, \succ_S, C)$, the outcome*

$$O' = (M, L \cap M[R], R, \succ_S, C)$$

obtained from O by removing reserve seats from every school not assigned a reserve-eligible student is stable.

Proof. Indeed, assume for sake of deriving a contradiction that (s, c) is a blocking pair in O' but not in O; As s's preferences are unchanged, it must be the case that $M[c]$ has higher priority than s under reserve allocation L, but s has higher priority than $M[c]$ under reserve allocation L'. Hence, the priority order of c must be different under reserve allocation L' than under reserve allocation L. It follows that $c \in (L \setminus L') \cup (L' \setminus L)$. As $L' = L \cap M[R]$, we must have that $c \in L \setminus M[R]$.

Hence $M[c]$ is not reserve-eligible, because $c \in L \setminus M[R]$. Therefore, if s had higher priority than $M[c]$ under reserve allocation L' (in which there's no reserve at c) s would have higher priority than $M[c]$ under reserve allocation L (in which there is a reserve at c)—a contradiction. $\square$

Lemma 2. *For all $R \subseteq S$, we have that*

$$\operatorname*{argmax}_{\mathcal{M}_F(\succ_S, C, R)} W_{\blacktriangleright, \succ_S} = \operatorname*{argmax}_{\mathcal{M}_A(\succ_S, C, R)} W_{\blacktriangleright, \succ_S} = \operatorname*{argmax}_{\mathcal{M}_F^G(\succ_S, C, R)} W_{\blacktriangleright, \succ_S} = \operatorname*{argmax}_{\mathcal{M}_A^G(\succ_S, C, R)} W_{\blacktriangleright, \succ_S}.$$

Proof. Consider the following function on outcomes. Take

$$f(M, L, R, \succ_S, C) = (\mathcal{DA}(L \cap M[R], R, \succ_S, C), L \cap M[R], R, \succ_S, C)$$

which essentially takes out reserves which are given to non-reserve eligible students and runs deferred acceptance.

Claim 1. *For all stable outcomes $O = (M, L, R, \succ_S, C)$, we have that*

$$W_{\blacktriangleright, \succ_S}(f(O)_M) \geq W_{\blacktriangleright, \succ_S}(M)$$

Proof. Consider the reserve allocation $L' = L \cap M[R]$. Let

$$O' = (M, L', R, \succ_S, C).$$

We now observe that O' is stable by Lemma 1. Note that

$$f(O) = (\mathcal{DA}(L', R, \succ_S, C), L', R, \succ_S, C)$$

is also stable, as the outcome of deferred acceptance. Therefore, each student receives a weakly better assignment in $f(O)_M$ to O'_M by optimality of deferred acceptance. Since, the welfare function is monotonic in student's preferences, we may conclude. □

The following claim is a consequence of Claim 1.

Claim 2. *For all outcomes $O = (M, L, R, \succ_S, C)$ with $M = \operatorname{argmax}_{\mathcal{M}_F(\succ_S, C, R)} W_{\blacktriangleright, \succ_S}$, we have that $f(O)_M = M$.*

Proof. Observe first that $f(O)_M \in \mathcal{M}_F(\succ_S, C, R)$, since deferred acceptance is stable and $|L \cap M[R]| \leq |L| \leq T$.

Now, by Claim 1 we have that

$$W_{\blacktriangleright, \succ_S}(f(O)_M) \geq W_{\blacktriangleright, \succ_S}(M)$$

But since $f(O)_M \in \mathcal{M}_F(\succ_S, C, R)$, as $O_M = \operatorname{argmax}_{\mathcal{M}_F(\succ_S, C, R)} W_{\blacktriangleright, \succ_S}$, we also have

$$W_{\blacktriangleright, \succ_S}(f(O)_M) \leq W_{\blacktriangleright, \succ_S}(M).$$

and we may conclude. □

Let $M = \operatorname{argmax}_{\mathcal{M}_F(\succ_S, C, R)} W_{\blacktriangleright, \succ_S}$. Consider a stable outcome $O' = (M, L', R, \succ_S, C)$. Claim 2 guarantees that $f(O)_M = M$. Let $O = f(O') = (M, L, R, \succ_S, C)$. As $L \subseteq M[R]$ and $M = \mathcal{DA}(L, R, \succ_S, C)$ by construction, the matching M is good attainable. Hence, we must have that

$$\max_{\mathcal{M}_F(\succ_S,C,R)} W_{\blacktriangleright,\succ_S} \leq \max_{\mathcal{M}_A(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}, \max_{\mathcal{M}_F^G(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}, \max_{\mathcal{M}_A^G(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}.$$

As $\mathcal{M}_F(\succ_S, C, R) \supseteq \mathcal{M}_A(\succ_S, C, R), \mathcal{M}_F^G(\succ_S, C, R), \mathcal{M}_A^G(\succ_S, C, R)$, we have that

$$\max_{\mathcal{M}_F(\succ_S,C,R)} W_{\blacktriangleright,\succ_S} \geq \max_{\mathcal{M}_A(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}, \max_{\mathcal{M}_F^G(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}, \max_{\mathcal{M}_A^G(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}.$$

Hence, we must have that

$$\max_{\mathcal{M}_F(\succ_S,C,R)} W_{\blacktriangleright,\succ_S} = \max_{\mathcal{M}_A(\succ_S,C,R)} W_{\blacktriangleright,\succ_S} = \max_{\mathcal{M}_F^G(\succ_S,C,R)} W_{\blacktriangleright,\succ_S} = \max_{\mathcal{M}_A^G(\succ_S,C,R)} W_{\blacktriangleright,\succ_S}.$$

As the function $W_{\blacktriangleright,\succ_S}$ is injective on matchings, the lemma follows. □

Definition 4. The *k-restriction* of a matching M is given by

$$M|_k = \bigcup_{1\leq i\leq k,\, M[\blacktriangleright_i]\neq\varnothing} \{(\blacktriangleright_i, M[\blacktriangleright_i])\}$$

Define the set of *k-restrictions* of a set of matchings S to be $S|_k$.

The following lemma allows us to formalize the notion of a student being irrelevant to the final outcome.

Lemma 3 (Low-Priority Lemma). *We have that*

$$\mathcal{DA}(L, \mathrm{Pre}_k, \succ_{S\setminus\{s_0\}}, C)|_k = \mathcal{DA}(L, \mathrm{Pre}_k, \succ_S, C)|_k$$

if $\blacktriangleright_k \sqsupset^0 s_0$.

Proof. The key step is to use order-independence of proposals in deferred acceptance [15]. Consider the two-stage process given below which computes $\mathcal{DA}(L, \mathrm{Pre}_k, \succ_S, C)$.

1. Have every student except for s_0 propose in order until a stable matching is reached.
2. Have student s_0 propose, and observe that one of two things can happen when he proposes to a school.
   (a) s_0 is rejected, so he proposes again.
   (b) s_0 takes the place of the student in the seat, who proposes next.

The result of the first step is $\mathcal{DA}(L, \mathrm{Pre}_k, \succ_{S\setminus\{s_0\}}, C)$. Let the students who are displaced over the course of the second step be, in order, $s_0, s_1, \ldots, s_k$. Therefore, for all $0 \leq i \leq k-1$, s_i must be preferred to s_{i+1} at some school. To show the result, it suffices to show that none of s_i can be reserve-eligible. To see this, assume for the sake of contradiction that s_t is a student in this order who is reserve-eligible. Observe that s_0 must be preferred to s_t at some school, but then

$$s_t \sqsupseteq^x \blacktriangleright_k \sqsupset^x s_0$$

for all nonnegative x and we may conclude. □

We can easily extend this lemma as follows.

Corollary 3. For a set T, if for each $s_0 \in T$, $\blacktriangleright_k \sqsupset^0 s_0$, then we have

$$\mathcal{DA}(L, \mathrm{Pre}_k, \succ_{S \smallsetminus T}, C)|_k = \mathcal{DA}(L, \mathrm{Pre}_k, \succ_S, C)|_k.$$

The following is a standard lemma, and was proven in [16].

Lemma 4 (Comparative Static, see [16])**.** *For any s_0, we have that*

$$\mathcal{DA}(L, R, \succ_{S\setminus\{s_0\}}, C)[s_1] \succeq_{s_1} \mathcal{DA}(L, R, \succ_S, C)[s_1]$$

for $s_1 \in S \setminus \{s_0\}$.

Appendix A.3. Proof of Proposition 1

We show a lemma that lays the groundwork for the rest of the proof. The key idea is to interpret opt$[i]$ as a function that takes an optimal matching from a set of matchings, and showing that one of these sets is a subset of the other.

Lemma 5. *We have $W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i+1]) \geq W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i])$.*

Claim 3. *If $M \in \mathcal{M}_F^G(\succ_S, C, \mathrm{Pre}_i)$, then $M \in \mathcal{M}_F^G(\succ_S, C, \mathrm{Pre}_{i+1})$*

Proof of Claim 3. To prove this, it suffices to show that if $O_1 = (M, \mathrm{Pre}_i, \succ_S, C)$ is a good T-feasible partial outcome, then $O_2 = (M, \mathrm{Pre}_{i+1}, \succ_S, C)$ is also a good T-feasible partial outcome. By the hypothesis, there exists a stable outcome $(M, L, \mathrm{Pre}_i, \succ_S, C)$ with $|L| \leq T$. We claim that $(M, L, \mathrm{Pre}_{i+1}, \succ_S, C)$ is also a stable outcome. Goodness follows directly, since we have that $L \subseteq M[\mathrm{Pre}_i] \subseteq M[\mathrm{Pre}_{i+1}]$. T-feasibility also follows, since $|L| \leq T$. Stability follows by goodness; if (s, c) blocks O_2, then $c \succ_s M[s]$ and $s \succ_c M[c]$. If (s, c) is not to block O_1, then the preferences don't change, so we must have $M[c] \succ_c s$ in O_1. The priority changing implies that c is a reserve seat, and that student s and $M[c]$ change order when $\blacktriangleright_{i+1}$ becomes reserve-eligible. Therefore, we have that s is $\blacktriangleright_{i+1}$, and $M[c]$ cannot be reserve-eligible. However, if $M[c]$ isn't reserve-eligible, then by goodness there is no reserve seat at c, and we may conclude. □

We now use this claim to prove the lemma.

Proof of Lemma 5. From Lemma 2,

$$\mathrm{opt}[i] = \operatorname*{argmax}_{\mathcal{M}_F(\succ_S, C, \mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S} = \operatorname*{argmax}_{\mathcal{M}_F^G(\succ_S, C, \mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S}$$

Now the result follows from Claim 3. □

We can now prove Proposition 1.

Proof of Proposition 1. By definition, there exists a reserve distribution L_0 with size at most T such that $(\mathrm{opt}[i+1], L_0, \mathrm{Pre}_{i+1}, \succ_S, C)$ is a stable outcome.

Define $M_0 = \mathcal{DA}(L_0, \mathrm{Pre}_i, \succ_S, C)$. Note that

$$\begin{aligned} M_0[\blacktriangleright_k] &= \mathcal{DA}(L_0, \mathrm{Pre}_i, \succ_S, C)[\blacktriangleright_k] \\ &= \mathcal{DA}(L_0, \mathrm{Pre}_i, \succ_{S\setminus\{\blacktriangleright_{i+1}\}}, C)[\blacktriangleright_k] \\ &= \mathcal{DA}(L_0, \mathrm{Pre}_{i+1}, \succ_{S\setminus\{\blacktriangleright_{i+1}\}}, C)[\blacktriangleright_k] \\ &\succeq_{\blacktriangleright_k} \mathcal{DA}(L_0, \mathrm{Pre}_{i+1}, \succ_S, C)[\blacktriangleright_k] \\ &= \mathrm{opt}[i+1][\blacktriangleright_k] \end{aligned}$$

for $1 \leq k \leq i$, where the second equality follows from Lemma 3 and the inequality follows from Lemma 4.

By Lemma 5 and by the definition of opt$[i]$,

$$W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i+1]) \geq W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i]) \geq W_{\blacktriangleright,\succ_S}(M_0).$$

Assume for the sake of contradiction that for some $j \leq i$, $\mathrm{opt}[i+1][\blacktriangleright_j] \neq M_0[\blacktriangleright_j]$. Take the smallest such j, and observe that since $M_0[\blacktriangleright_j] \succ_{\blacktriangleright_j} \mathrm{opt}[i+1][\blacktriangleright_j]$, $W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i+1]) \leq W_{\blacktriangleright,\succ_S}(M_0)$ which is contradiction. Thus, $\mathrm{opt}[i+1][\blacktriangleright_j] = M_0[\blacktriangleright_j]$ for $j \leq i$. Assume for the sake of contradiction that for some $j \leq i$, $\mathrm{opt}[i+1][\blacktriangleright_j] \neq \mathrm{opt}[i][\blacktriangleright_j]$. By lexicographic comparisons, it must be true that $\mathrm{opt}[i+1][\blacktriangleright_j] \succ_{\blacktriangleright_j} \mathrm{opt}[i][\blacktriangleright_j]$. But then,

$$M_0[\blacktriangleright_j] = \mathrm{opt}[i+1][\blacktriangleright_j] \succ_{\blacktriangleright_j} \mathrm{opt}[i][\blacktriangleright_j],$$

which contradicts $W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i]) \geq W_{\blacktriangleright,\succ_S}(M_0)$, so $\mathrm{opt}[i+1][\blacktriangleright_j] = \mathrm{opt}[i][\blacktriangleright_j]$ for $j \leq i$ and we may conclude. □

Appendix A.4. Proof of Proposition 2

It suffices to show that $\mathrm{dp}[i][\blacktriangleright_j] = \mathrm{opt}[i][\blacktriangleright_j]$ for $1 \leq j \leq i$, because the welfare-maximizing stable matching for the rest of the students (whose precedence is lower than all students in Pre_i) is a serial dictatorship indexed by test score, as none of the students are reserve-eligible.

Suppose otherwise, and that for some i, $\mathrm{dp}[i] \neq \mathrm{opt}[i]$. Then there must be a minimal j such that $\mathrm{dp}[j] \neq \mathrm{opt}[j]$. By minimality, $\mathrm{dp}[j-1] = \mathrm{opt}[j-1]$. By Proposition 1 and by construction, for $1 \leq k \leq j-1$, $\mathrm{dp}[j][\blacktriangleright_k] = \mathrm{dp}[j-1][\blacktriangleright_k] = \mathrm{opt}[j-1][\blacktriangleright_k] = \mathrm{opt}[j][\blacktriangleright_k]$. Therefore, $\mathrm{dp}[j][\blacktriangleright_j] \neq \mathrm{opt}[j][\blacktriangleright_j]$. However, since $\mathrm{dp}[j]$ is T-feasible this implies that $\mathrm{opt}[j][\blacktriangleright_j] \succ_{\blacktriangleright_j} \mathrm{dp}[j][\blacktriangleright_j]$, and thus that the minimum feasible number of $(M' \cup \{(\blacktriangleright_j, \mathrm{opt}[j][\blacktriangleright_j])\}, \mathrm{Pre}_j, \succ_{\mathrm{Pre}_j}, \succ_{\mathrm{opt}[j][\blacktriangleright_j] \cup M'[\mathrm{Pre}_{j-1}]})$ is at most T, since the matching $\mathrm{opt}[j]$ is T-feasible, which is contradiction.

Appendix A.5. Proof of Proposition 3

Now we show that DPRA is strategy-proof for targeted students. The intuitive reason for this is that DPRA essentially only queries for the preferences of the student $\blacktriangleright_i$ when computing $\mathrm{dp}[i]$.

Note that $\blacktriangleright_i$'s assignment is fixed following step i. We claim that $\blacktriangleright_i$'s preferences cannot affect the assignments of Pre_{i-1}.

Lemma 6. *If $k < i$, then*

$$\operatorname*{argmax}_{\mathcal{M}_F(\succ_{S\setminus\{\blacktriangleright_i\}},C,\mathrm{Pre}_k)|_k} W_{\blacktriangleright,\succ_S} = \operatorname*{argmax}_{\mathcal{M}_F(\succ_S,C,\mathrm{Pre}_k)|_k} W_{\blacktriangleright,\succ_S}$$

Claim 4. *Removing $\blacktriangleright_i$ preserves the set of T-attainable restricted k-matchings if $k < i$, that is*

$$\mathcal{M}_A(\succ_{S\setminus\{\blacktriangleright_i\}}, C, \mathrm{Pre}_k)|_k = \mathcal{M}_A(\succ_S, C, \mathrm{Pre}_k)|_k.$$

Proof. Note that $M \in \mathcal{M}_A(\succ_{S\setminus\{\blacktriangleright_i\}}, C, \mathrm{Pre}_k)|_k$ if and only if there exists $L \subseteq S$ with $|L| \leq T$ and $M = \mathcal{DA}(L, \mathrm{Pre}_k, \succ_{S\setminus\{\blacktriangleright_i\}}, C)|_k$. By Lemma 3, the second hypothesis holds if and only if there exists $L \subseteq S$ with $|L| \leq T$ and $M = \mathcal{DA}(L, \mathrm{Pre}_k, \succ_{S\setminus\{\blacktriangleright_i\}}, C)|_k$—i.e., if and only if $M \in \mathcal{M}_A(\succ_S, C, \mathrm{Pre}_k)|_k$. The claim follows. □

Proof of Lemma 6. We have that

$$\operatorname*{argmax}_{\mathcal{M}_F(\succ_{S\setminus\{\blacktriangleright_i\}},C,\mathrm{Pre}_k)|_k} W_{\blacktriangleright,\succ_S} = \operatorname*{argmax}_{\mathcal{M}_A(\succ_{S\setminus\{\blacktriangleright_i\}},C,\mathrm{Pre}_k)|_k} W_{\blacktriangleright,\succ_S} = \operatorname*{argmax}_{\mathcal{M}_A(\succ_S,C,\mathrm{Pre}_k)|_k} W_{\blacktriangleright,\succ_S} = \operatorname*{argmax}_{\mathcal{M}_F(\succ_S,C,\mathrm{Pre}_k)|_k} W_{\blacktriangleright,\succ_S}$$

where the first and third equalities follow from Lemma 2, and the second equality follows from Claim 4. □

Denoting $\succ'_S = (\succ'_{\blacktriangleright_i}, \succ_{S\setminus\{\blacktriangleright_i\}})$, assume for contradiction that

$$\left(\operatorname*{argmax}_{\mathcal{M}^G_F(\succ'_S,C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ'_S}\right)[\blacktriangleright_i] \succ_{\blacktriangleright_i} \left(\operatorname*{argmax}_{\mathcal{M}^G_F(\succ_S,C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S}\right)[\blacktriangleright_i]$$

From Lemma 6, we have that

$$\left(\operatorname*{argmax}_{\mathcal{M}^G_F(\succ'_S,C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ'_S}\right)\Bigg|_{i-1} = \left(\operatorname*{argmax}_{\mathcal{M}^G_F(\succ_{S\setminus\{\blacktriangleright_i\}},C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S}\right)\Bigg|_{i-1} = \left(\operatorname*{argmax}_{\mathcal{M}^G_F(\succ_S,C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S}\right)\Bigg|_{i-1}$$

Combined with the above, this implies that

$$\max_{\mathcal{M}^G_F(\succ'_S,C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S} > \max_{\mathcal{M}^G_F(\succ_S,C,\mathrm{Pre}_i)} W_{\blacktriangleright,\succ_S}.$$

Let the optimal matching in $M^G_F(\succ'_S, C, \mathrm{Pre}_i)$ be part of the following outcome:

$$O_1 = (M, L, \mathrm{Pre}_i, \succ'_S, C)$$

where $|L| \leq T$. By optimality, it must the case that

$$O'_1 = (M, L, \mathrm{Pre}_i, \succ_S, C)$$

is not stable or not good. O'_1 is clearly good, so it must not be stable. The only student who can form blocking pairs is $\blacktriangleright_i$, since priorities do not change and only preferences do.

Assume that he forms blocking pairs with schools

$$c_{q_1} \succ_{\blacktriangleright_i} c_{q_2} \succ_{\blacktriangleright_i} \dots \succ_{\blacktriangleright_i} c_{q_k},$$

so $c_{q_j} \succ_{\blacktriangleright_i} M[\blacktriangleright_i]$ for all j. Observe that $c_{q_1} \notin L$, since s' cannot be reserve-eligible by virtue of being blocked by $\blacktriangleright_i$ and O_1' is good. Therefore, since the seat is not reserve, s' has a strictly lower test score than $\blacktriangleright_i$.

Now denote S'' as the set of students with strictly lower test scores than $\blacktriangleright_i$. Consider the matching

$$M' = M|_{S \setminus (S'' \cup \{\blacktriangleright_i\})} \cup \{(\blacktriangleright_i, c_{q_1})\}$$

where we replace $\blacktriangleright_i$'s assignment with c_{q_1}, and remove students with test scores lower than $\blacktriangleright_i$.

Claim 5. *The outcome*

$$(M', L, \mathrm{Pre}_i, \succ_{S \setminus S''}, C)$$

is stable.

Proof of Claim 5. Assume that (s, c) forms a blocking pair in this outcome. Then we have $s \succ_c M'[c]$ and $c \succ_s M'[s]$. However, by stability of O_1, we have that $M[c] \succeq_c s$ or $M[s] \succeq_s c$. The latter is impossible since by construction, all students prefer their old assignment to their new one. Since all schools have an improvement in their assignments other than $M[\blacktriangleright_i]$, so $M[\blacktriangleright_i]$ must be part of a blocking pair. However, now we have contradiction since for any student $s \neq \blacktriangleright_i$, $s \succ_c \blacktriangleright_i$ so we must have $M[s] \succeq_s M[\blacktriangleright_i]$. But then we have $M'[s] \succeq_s M[\blacktriangleright_i]$, contradicting $c \succ_s M'[s]$, and we are done. $\square$

Now consider $\mathcal{DA}(L, \mathrm{Pre}_i, \succ_{S \setminus S''}, C)$, which will improve upon M'. By Corollary 3, we have that

$$\mathcal{DA}(\succ_{S \setminus S''}, C, L, \mathrm{Pre}_i)[\blacktriangleright_j] = \mathcal{DA}(L, \mathrm{Pre}_i, \succ_S, C)[\blacktriangleright_j]$$

for $j \leq i$.

Therefore,

$$\begin{aligned}
\mathcal{DA}(L, \mathrm{Pre}_i, \succ_S, C)[\blacktriangleright_i] &= \mathcal{DA}(L, \mathrm{Pre}_i, \succ_{S \setminus S''}, C)[\blacktriangleright_i] \\
&\succeq_{\blacktriangleright_i} c_{i_r} \\
&\succ_{\blacktriangleright_i} M[\blacktriangleright_i] \\
&\succ_{\blacktriangleright_i} \mathrm{opt}[i][\blacktriangleright_i]
\end{aligned}$$

and for all $j < i$,

$$\begin{aligned}
\mathcal{DA}(L, \mathrm{Pre}_i, \succ_S, C)[\blacktriangleright_j] &= \mathcal{DA}(L, \mathrm{Pre}_i, \succ_{S \setminus S''}, C)[\blacktriangleright_j] \\
&\succeq_{\blacktriangleright_j} M'[\blacktriangleright_j] \\
&= M[\blacktriangleright_j] \\
&= \mathrm{opt}[i][\blacktriangleright_j]
\end{aligned}$$

The matching $\mathcal{DA}(L, \mathrm{Pre}_i, \succ_S, C)$ is T-feasible by construction. It must be true that

$$W_{\blacktriangleright,\succ_S}(\mathcal{DA}(L, \mathrm{Pre}_i, \succ_S, C)) \leq W_{\blacktriangleright,\succ_S}(\mathrm{opt}[i])$$

by definition as the first matching is T-feasible whereas the second matching optimizes over all T-feasible matchings, but the above two inequalities contradict this, and we may conclude.

Appendix A.6. Proof of Proposition 4

It suffices to show the result with an example. Assume that there are three students, three schools, two reserve-eligible students (s_1 and s_2) and one reserve seat. Let $x_c \equiv \infty$.

- The common priority ordering over all schools is

$$s_3 \succ_{c_i} s_1 \succ_{c_i} s_2.$$

- The preferences by students are

$$\begin{aligned} s_1 &: c_1 \succ_{s_1} c_2 \succ_{s_1} c_3 \\ s_2 &: c_3 \succ_{s_2} c_1 \succ_{s_2} c_2 \\ s_3 &: c_3 \succ_{s_3} c_2 \succ_{s_3} c_1 \end{aligned}$$

With these preferences, s_2 will be matched to c_3, and s_3 will not be matched to c_3. s_3 can't be matched to c_3 since s_1 doesn't require a reserve. However, if we have s_3 submit false preferences $c_1 \succ_{s_3} c_3 \succ_{s_3} c_2$, the reserve seat will be used at c_1 to protect s_1's claim, and s_2 will not have a reserve seat. As a result, s_3 will be matched to c_3, so the mechanism is not strategy proof for non-reserve eligible students.

Appendix A.7. Proof of Proposition 5

The key step is to show that the matching underlying any stable outcome that includes M must be precisely the auxiliary matching M'. We then show that the BPA algorithm finds the smallest set of reserve locations to make M' stable.

Lemma 7. *Let*

$$O_0 = (M_0, L_0, R_0, \succ_S, C)$$

be a stable outcome. Let $S' \subseteq S$ and $C' \subseteq C$ be sets, and let $M_0' \subseteq M_0 \cap (S' \times C')$ be a matching. If $M_0 \setminus M_0' \subseteq (S \setminus S') \times (C \setminus C')$, then

$$O_0' = (M_0', L_0 \cap C' \cap M_0'[R_0 \cap S'], R_0 \cap S', \succ_{S'}, C')$$

is a stable outcome.

Proof. We first show that $O_0'' = (M_0', L_0 \cap C', R_0 \cap S', \succ_{S'}, C')$ is stable. The preferences (resp. priorities) of S' (resp. C') over the schools in C' (resp. the students in S') are the same in O_0 and O_0''. Note that $M_0'[x] = M_0[x]$ for all $x \in S' \cup C'$. Therefore, since the matching, preferences, and priorities are the same between O_0 and O_0'', (s, c) blocks O_0'' iff it blocks O_0 as well, and thus O_0'' is stable.

The conclusion of the lemma follows by Lemma 1. □

Lemma 8. *Assume that the outcomes* $(M_0, L_1, R_1, \succ_{S'}, C')$ *and* $(M_0', L_2, R_2, \succ_{S'}, C')$ *are both stable outcomes for* $S' \subseteq S$ *and* $C' \subseteq C$ *and* $R_1, R_2 \subseteq S$ *where at most one of* L_1 *and* R_1 *are nonempty, and at most one of* L_2 *and* R_2 *are nonempty. Then* $M_0 = M_0'$.

Proof. By [17, Proposition 6], M and M' must both be the result of serial dictatorship with order dictated by $\blacktriangleright$. Hence, M must equal M'. $\square$

Suppose that there exists $L \subset C$ and a matching M'' such that $(M'', L, R, \succ_S, C)$ is a stable outcome, and $M''[s] = M[s]$ for $s \in S'$ and $M''[c] = M[c]$ for $c \in C'$. By Lemma 7, the outcome

$$(M'' \setminus M, L', \varnothing, \succ_{S \setminus S'}, C \setminus C')$$

is stable. The outcome

$$(\mathcal{DA}(\varnothing, \varnothing, \succ_{S \setminus S'}, C \setminus C'), \varnothing, \varnothing, \succ_{S \setminus S'}, C \setminus C')$$

is also stable, so by Lemma 8, we must have that $M'' \setminus M = \mathcal{DA}(\varnothing, \varnothing, \succ_{S \setminus S'}, C \setminus C')$ and hence that $M'' = M'$.

Claim 6. *Consider the outcomes* $O_0 = (M', L, R, \succ_S, C)$ *and* $O_1 = (M', L \cup \{c_0\}, R, \succ_S, C)$.

(a) *If* $c_0 \in M'[R]$, O_1 *is stable if* O_0 *is stable.*
(b) *If* $c_0 \notin M'[R]$, O_0 *is stable if* O_1 *is stable.*

Proof. We prove the contrapositives of assertions (a) and (b) separately.

(a) Let (s, c) be a blocking pair in O_1; we show that (s, c) is a blocking pair in O_0. If $c \neq c_0$, then the result is immediate, as such c have the same priorities in both O_0 and O_1. Hence, it suffices to consider the case in which $c = c_0$.
If (s, c_0) does not block O_0, we must have that c_0 prioritizes $M[c_0]$ over s in O_0, but c_0 prioritizes s over $M[c_0]$ in O_1. However, since $M[c_0] \in R$, adding a reserve at c_0 will never reduce $M[c_0]$'s priority relative to any other student—including s. But this would mean that (s, c_0) also could not block O_1, a contradiction.
(b) Following a similar argument to that in part (a), we see that it suffices to consider the case in which $c = c_0$.
Now, if (s, c_0) is not a blocking pair in O_1, then since s's preferences do not change between O_1 and O_0, it must be the case that either (i) c_0 prioritizes $M[c_0]$ over s in O_1, but c_0 prioritizes s over $M[c_0]$ in O_0, or (ii) s is acceptable to c_0 in O_1, but not in O_0. Case (ii) is ruled out by our assumption that all students are acceptable to all schools. Therefore, we must have case (i). But since $M[c_0]$ is a non-reserve-eligible student, adding a reserve at c_0 will never raise her priority with respect to another student—meaning that (s, c_0) cannot a block O_0, a contradiction. $\square$

Applying Claim 6 repeatedly yields the following result.

Claim 7. *If there exists an* $L \subseteq C$ *such that* $(M', L, R, \succ_S, C)$ *is a stable outcome, then* $(M', M'[R], R, \succ_S, C)$ *is also stable.*

The BPA returns ∞ in the case where $(M', M'[R], R, \succ_S, C)$ is not stable. Indeed, if $(M', M'[R], R, \succ_S, C)$ is not stable, then by Claim 7 there is no L such that $(M', L, R, \succ_S, C)$ is stable and we may conclude.

Otherwise, there is at least one set L such that $(M, L, R, \succ_S, C)$ is stable; $|L_0|$ is the minimum feasible number of O. Let L_0 be the smallest subset of C so that $(M', L_0, R, \succ_S, C)$ is stable. Let U be the set of schools in $M'[R]$ that are blocked by students in $S \setminus R$ in the outcome $(M', \varnothing, R, \succ_S, C)$. Since BPA outputs $|U|$, to show that BPA outputs the minimum feasible number $|L_0|$, it suffices to show that $L_0 = U$.

We first show that $U \subseteq L_0$. Assume for the sake of deriving a contradiction that there exists $c \in M'[R]$ such that $c \in U$, but $c \notin L_0$. This means there is a student $s \in S \setminus R$ so that (s, c) blocks the outcome $(M', \varnothing, R, \succ_S, C)$. But then, (s, c) also blocks the outcome $(M', L_0, R, \succ_S, C)$, since the priority of c is the same in $(M', \varnothing, R, \succ_S, C)$ and $(M', L_0, R, \succ_S, C)$, which is a contradiction.

We next show that $L_0 \subseteq U$. By Lemma 1, the outcome $(M'[R], L_0 \cap M'[R], R, \succ_S, C)$ is stable as well, and $L_0 \cap M'[R]$ has smaller size than L_0. Hence, the minimality of L_0 entails that $L_0 \subseteq M'[R]$. Assume for the sake of deriving a contradiction that there exists $c \in M'[R]$ so that $c \in L_0 \setminus U$. This means there are no students $s \in S \setminus R$ so that (s, c) forms a blocking pair in the outcome $(M', \varnothing, R, \succ_S, C)$. Then we claim that $(M', L_0 \setminus \{c\}, R, \succ_S, C)$ is also stable, contradicting minimality of L_0. Assume that $(M', L_0 \setminus \{c\}, R, \succ_S, C)$ is not stable; this implies that there exists a student s so that (s, c) forms a blocking pair in this outcome. Take any such student s_0 so that (s_0, c) blocks the outcome $(M', L_0 \setminus \{c\}, R, \succ_S, C)$. If $s_0 \in R$, note that (s_0, c) also blocks $(M', L_0, R, \succ_S, C)$. Therefore, $s_0 \in S \setminus R$. But then, (s_0, c) forms a blocking pair in $(M', \varnothing, R, \succ_S, C)$, which implies that $c \in U$ and is contradiction.

Since $U \subseteq L_0$ and $L_0 \subseteq U$, we have that $L_0 = U$. As $|U|$ is the output of the BPA, outputting $|U|$ gives us $|L_0|$ and we are done.

Appendix A.8. Proof of Proposition 6

Note that each step of dynamic programming (of which there are b) runs deferred acceptance once (at $\mathcal{O}(mn)$ time complexity) and *BPA* up to m times (at $\mathcal{O}(mn)$ time complexity each), so the overall runtime is bounded by $b \cdot mn + b \cdot mn \cdot m = \mathcal{O}(bm^2n)$.