A Constant Bound for the Periods of Parallel Chip-firing Games with Many Chips

Paul Myer Kominers and Scott Duke Kominers

Abstract. We prove that any parallel chip-firing game on a graph G with at least $4|E(G)| - |V(G)|$ chips stabilizes, i.e. such a game has eventual period of length 1. Furthermore, we obtain a polynomial bound on the number of rounds before stabilization. This result is a counterpoint to previous results which showed that the eventual periods of parallel chip-firing games with few chips need not even be polynomially bounded.

Mathematics Subject Classification (2000). 05C35, 05C85, 68Q25 (Primary); 37B15, 68R10, 68Q80 (Secondary).

Keywords. chip-firing, parallel chip-firing, stabilization, polynomial time.

1. Introduction

We let G be a finite, undirected, connected graph and denote the vertex and edge sets of G by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, we denote the degree of v by $\deg(v)$.

In a *chip-firing game on* G, some number of chips are distributed among the $|V(G)|$ vertices of G. Then, in each of a sequence of rounds $t = 1, 2, \ldots$, a vertex $v \in V$ with more than $\deg(v)$ chips is selected and *fired*—one chip from v is moved to each of v's neighbors. The *parallel chip-firing game on* G is defined similarly: chips are distributed among the $|V(G)|$ vertices of G, and in each of a sequence of rounds $t = 1, 2, \ldots$, *all* vertices $v \in V$ with more than $\deg(v)$ chips are fired.

The chip-firing game was first introduced by Spencer [7] for infinite graphs; Björner, Lovász, and Shor [3] extended the game's definition to general finite graphs. Terminating chip-firing games on undirected graphs have been studied

The second author gratefully acknowledges the support of a Harvard Mathematics Department Highbridge Fellowship.

extensively and are surprisingly well-behaved. For example, the length of a terminating chip-firing game on a graph G is bounded by a polynomial in the characteristics of G (see [8]). Chip-firing games also have important applications; notably, they are related to Tutte polynomials (see [6]) and the critical groups of graphs (see [1]).

Bitar and Goles [2] introduced the parallel chip-firing game, observing that such a game must necessarily converge towards a periodic sequence of chip configurations. They furthermore proved that any parallel chip-firing game on a finite, undirected, connected, acyclic graph has eventual period of length at most 2. Disproving two previous conjectures to the contrary, Kiwi *et al.* [4] later showed that the periods of parallel chip-firing games need not be polynomially bounded. Specifically, Kiwi *et al.* [4] constructed a parallel chip-firing game on an n-node, connected, undirected graph having eventual period of length $\exp(\Omega(\sqrt{n \log n}))$.

In this note, we show the following counterpoint to the result of Kiwi *et al.* [4].

Theorem 1. *If G is a finite, undirected, connected graph, then any parallel chip-firing game on G with at least $4|E(G)| - |V(G)|$ chips has eventual period of length* 1.

We show additionally that when the conditions of Theorem 1 hold, the game on G converges to its period-1 chip configuration in a number of rounds bounded above by a polynomial in the characteristics of G.

Our approach draws from the literature on chip-firing, using in particular a key result from Tardos's [8] proof that terminating chip-firing games conclude in polynomial time.

2. The Setting

We denote by $\varphi_t(v)$ the total number times a vertex $v \in V(G)$ has fired by the end of round t. For consistency, we throughout denote by c the total number of chips in a parallel chip-firing game on G.

We say that a vertex has *stabilized* in some round if, after that round, the number of chips on that vertex will not change throughout the remainder of the game. Thus, a parallel chip-firing game on G has eventual period 1 if and only if all vertices of G stabilize. Abusing terminology slightly, we therefore say that a parallel chip-firing game on G *stabilizes* if its eventual period is 1.

We say that a vertex $v \in V(G)$ is *abundant* if it holds at least $2\deg(v)$ chips. Any vertex $v \in V(G)$ with $k \geq \deg(v)$ chips at the beginning of a round passes $\deg(v)$ chips to its neighbors and can, at most, receive one chip from each of its $\deg(v)$ neighbors. Thus, such a vertex cannot end the round with more than k chips. In particular, the set of abundant vertices of G can only shrink over the course of a parallel chip-firing game on G.

3. Main Theorem

We now prove the following stabilization theorem.

Theorem 2. *Let G be a finite, undirected, connected graph with diameter d. In any parallel chip-firing game on G with*

$$c \geq 4|E(G)| - |V(G)|$$

chips, every vertex $v \in V(G)$ will stabilize within $|V(G)| \cdot d \cdot c$ rounds.

This result implies Theorem 1 of the introduction. Additionally, Theorem 2 encapsulates the $c \geq 3n$ cases of the result of the first author [5] for parallel chip-firing games on n-cycles. Our methods are inspired by those of Tardos [8]; they are essentially independent of the arguments used in [5].

We use the following lemma, which is a special case of Lemma 5 of Tardos [8].

Lemma 3. *Let $v, v' \in V(G)$ be adjacent vertices of G. Then, $|\varphi_t(v) - \varphi_t(v')| \leq c$ for all t.*

Additionally, we need an observation about the condition $c \geq 4|E(G)| - |V(G)|$.

Lemma 4. *For G a graph and $c \geq 4|E(G)| - |V(G)|$, in any parallel chip-firing game on G with c chips there is at least one vertex $v_* \in V(G)$ which fires every round.*

Proof. It suffices to find a vertex $v_* \in V(G)$ which fires every round t during which some vertex $v \in V(G)$ holds fewer than $2\deg(v) - 1$ chips.

As observed above, it is not possible for a vertex $v \in V(G)$ which is not abundant at the beginning of round t to become abundant after round t. However, the condition

$$c \geq 4|E(G)| - |V(G)|$$

guarantees that whenever some $v \in V(G)$ holds fewer than $2\deg(v) - 1$ chips there is also at least one abundant vertex $v' \in V(G)$. The existence of some vertex $v_* \in V(G)$ which is abundant in every round when some vertex $v \in V(G)$ has fewer than $2\deg(v) - 1$ chips then follows immediately, and we have the lemma. ☐

We may now proceed with the proof of our main result.

Proof of Theorem 2. By Lemma 4, there is some vertex $v_* \in V(G)$ which fires every round. Denoting the rounds by $t = 1, 2, \ldots$, we then have $\varphi_t(v_*) = t$ for all rounds t. By Lemma 3, we then know that

$$|\varphi_t(v_*) - \varphi_t(v)| \leq d \cdot c$$

for all t and $v \in V(G)$. Since $\varphi_t(v_*)$ is strictly increasing in t, no $v \in V(G)$ may fail to fire for more than $d \cdot c$ rounds. In the worst case, all but one of the vertices fire in each round when some vertex does not fire; hence after $|V(G)| \cdot d \cdot c$ rounds all the vertices of G fire every round, and the game has stabilized. ☐

4. Remarks

4.1. On the Exponential Period Lengths Observed by Kiwi et al. [4]

The results of Kiwi *et al.* [4] show that parallel chip-firing games with sufficiently few chips can have eventual periods of exponential lengths. By contrast, our Theorem 1 shows that the period-lengths of parallel chip-firing games with sufficiently many chips are bounded by the constant 1.

The main class of examples provided by Kiwi *et al.* [4] uses graphs G_m with exactly $3\left(\sum_{i=1}^{m} p_i\right)+1$ vertices, $3\left(\sum_{i=1}^{m} p_i\right)+m$ edges, and $\underline{c}_m := 3\left(\sum_{i=1}^{m} p_i\right)+m$ chips. (Here, $m>1$ is fixed and $\{p_i\}_{i=1}^{\infty}$ is the ordered set of primes.) There is a substantial gap between $\underline{c}_m$ and

$$\bar{c}_m := 4\left(3\sum_{i=1}^{m} p_i + m\right) - \left(3\sum_{i=1}^{m} p_i + 1\right) = 9\sum_{i=1}^{m} p_i + 4m - 1,$$

the number of chips required for a parallel chip-firing game on G_m to be guaranteed to stabilize. The behavior of parallel chip-firing games on G_m with $\bar{c}_m > c > \underline{c}_m$ chips has not been studied; it would be interesting to know when a polynomially bounded period length can be guaranteed. In particular, it seems surprising that the period-length bound proceeds from exponential to constant pursuant to a *polynomial* increase in the number of chips.

4.2. On Possible Improvements of our Bounds

Our Lemma 4 is, in some sense, dual to Lemma 4 of Tardos [8] which shows that for any terminating chip-firing game on G there is a distinguished vertex $v_* \in V(G)$ which never fires. Such a result need not hold when $c < 4|E(G)| - |V(G)|$. Indeed, the two-vertex graph G with vertex set $V(G) = \{v_1, v_2\}$ and edge set $E(G) = \{\{v_1, v_2\}\}$ provides a simple counterexample. We have

$$4|E(G)| - |V(G)| - 1 = 4 \cdot 1 - 2 - 1 = 1,$$

while no vertex fires in every round of a single-chip parallel chip-firing game on G.

Nonetheless, the first author [5] has shown that the conclusion of Theorem 1 holds when G is an n-cycle and $c \geq 3n-2 = 4|E(G)| - |V(G)| - 2$. Thus, the lower bound on c used in Theorems 1 and 2 may be relaxed in special cases; it may be possible to relax this bound more generally.

Additionally, the "worst case" scenario examined in the proof of Theorem 2 seems unlikely to occur often when the number of chips on G is large. Thus, it is likely that the c-dependence of the stabilization-time bound in Theorem 2 can be relaxed.

References

[1] N. L. Biggs, Chip-firing and the critical group of a graph, Journal of Algebraic Combinatorics, **9**, 25–45 (1999).

[2] J. Bitar and E. Goles, Parallel chip firing games on graphs, Theoretical Computer Science, **92**, 291–300 (1992).

[3] A. Björner, L. Lovász, and P. Shor, Chip-firing games on graphs, European Journal of Combinatorics, **12**, 283–291 (1991).

[4] M. A. Kiwi et al., No polynomial bound for the period of the parallel chip firing game on graphs, Theoretical Computer Science, **136**, 527–532 (1994).

[5] P. M. Kominers, The candy-passing game for $c \geq 3n - 2$, Pi Mu Epsilon Journal, **12**, 459–460 (2008).

[6] C. M. López, Chip firing and the Tutte polynomial, Annals of Combinatorics, **1**, 253–259 (1997).

[7] J. Spencer, Balancing vectors in the max norm, Combinatorica, **6**, 55–65 (1986).

[8] G. Tardos, Polynomial bound for a chip firing game on graphs, SIAM Journal on Discrete Mathematics, **1**, 397–398 (1988).

Paul Myer Kominers
Department of Economics, Massachusetts Institute of Technology
c/o 8520 Burning Tree Road
Bethesda, MD 20817
e-mail: pkoms@mit.edu

Scott Duke Kominers
Department of Economics, Harvard University, and Harvard Business School
Wyss Hall, Harvard Business School
Soldiers Field, Boston, MA 02163
e-mail: kominers@fas.harvard.edu