FINDING MATRICES WHICH SATISFY FUNCTIONAL EQUATIONS

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Linear algebra is typically taught shortly after calculus. Thus, calculus-informed linear algebra problems offer an exceptional opportunity to illustrate interactions between different branches of undergraduate mathematics.

For example, we consider the following problem which (as we shall see) has its roots in calculus:

Problem. *Find a* 4×4*, nonsingular, nonconstant matrix function* $N(x)$ *which satisfies the functional equation*

$$N(2x) - (N(x))^8 = 0.$$

At first glance, this problem appears to be quite difficult. Beyond the likely difficulty of finding such a matrix $N(x)$, it is not even immediately clear how one would *prove* that a matrix $N(x)$ is actually a solution without a great deal of matrix algebra. However, this problem is not hard as it seems. In fact, it is one of a large class of problems which can be solved via a surprising method based upon single-variable calculus.

In this JOURNAL, Khan [2] used nilpotent matrices and Taylor series to find matrix functions satisfying the exponential functional equation, $f(x + y) = f(x) \cdot f(y)$. His method is an example of a much more general theory of matric power series due to Weyr [4], which can be used to find matrix functions satisfying a variety of functional equations. (Rinehart [3] gives an excellent survey of Weyr's approach. Higham [1, ch. 4] gives a more comprehensive account, as well as further applications.)

We say that a set of real-valued functions $\{f_i(x)\}_{i=1}^n \subset \mathcal{C}^\infty(\mathbb{R})$ *satisfies an analytic functional equation* E if there is an analytic function E such that

$$E(f_1, \ldots, f_n)(x) = 0$$

identically for all $x \in \mathbb{R}$. For example, the trigonometric functions $f_1 = \sin(x)$ and $f_2 = \cos(x)$ satisfy the analytic functional equation

$$E(f_1, f_2) = f_1^2 + f_2^2 - 1 \equiv 0.$$

Now, for any set of real-valued functions $\{f_i(x)\}_{i=1}^n \subset \mathcal{C}^\infty(\mathbb{R})$ satisfying the analytic functional equation E, we will find a set of associated matrix functions $\{A_i(x)\}_{i=1}^n$ satisfying the same functional equation E.

Approximating each f_i by its Taylor expansion about the origin, we obtain

$$f_i(x) = f_i(0)x^0 + \frac{f_i^{(1)}(0)x}{1!} + \frac{f_i^{(2)}(0)x^2}{2!} + \cdots + \frac{f_i^{(k)}(0)x^k}{k!} + \cdots,$$

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where $f_i^{(j)}$ is the j-th derivative of the function f_i. We let A be any nilpotent matrix with index of nilpotence k and then take

$$\begin{aligned} A_i(x) := f_i(Ax) &= f_i(0)I + \frac{f_i^{(1)}(0)Ax}{1!} + \frac{f_i^{(2)}(0)A^2x^2}{2!} + \cdots \\ &= f_i(0)I + \frac{f_i^{(1)}(0)Ax}{1!} + \frac{f_i^{(2)}(0)A^2x^2}{2!} + \cdots + \frac{f_i^{(k-1)}(0)A^{k-1}x^{k-1}}{(k-1)!}. \end{aligned}$$

If the functions $\{f_i(x)\}_{i=1}^n$ satisfy the analytic functional equation E, then the Taylor series of the f_i do as well, as E is continuous. Thus, the matrix functions $\{A_i(x)\}_{i=1}^n$ found from the Taylor series of the f_i must also satisfy the functionl equation E.

We begin with a simple example. For aesthetic reasons, we will work with the nilpotent matrix

$$A = \begin{pmatrix} 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 12 \\ 6 & 6 & 0 & 0 \\ -6 & -6 & 0 & 0 \end{pmatrix},$$

obtained by conjugating the Jordan-form nilpotent matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12 \\ -72 & -72 & 0 & 0 \\ 0 & 0 & -864 & -864 \end{pmatrix}.$$

We obtain from the Taylor series for $f_1 = \sin(x)$ and $f_2 = \cos(x)$ the matrices

$$\begin{aligned} A_1(x) = f_1(Ax) = Ax - \frac{A^3x^3}{6} &= \begin{pmatrix} 0 & 0 & 12x - 144x^3 & -144x^3 \\ 0 & 0 & 144x^3 & 144x^3 + 12x \\ 6x & 6x & 0 & 0 \\ -6x & -6x & 0 & 0 \end{pmatrix}, \\ A_2(x) = f_2(Ax) = I - \frac{A^2x^2}{2} &= \begin{pmatrix} 1 - 36x^2 & -36x^2 & 0 & 0 \\ 36x^2 & 36x^2 + 1 & 0 & 0 \\ 0 & 0 & 1 - 36x^2 & -36x^2 \\ 0 & 0 & 36x^2 & 36x^2 + 1 \end{pmatrix}. \end{aligned}$$

We have immediately from this construction that $A_1(x)$ and $A_2(x)$ commute. More interestingly, these matrix functions satisfy the trigonometric functional equations. We therefore find the familiar identity

$$(A_1(x))^2 + (A_2(x))^2 = I.$$

Similarly, we obtain analogues of the "double-angle" formulas,

$$\begin{aligned} A_1(2x) &= 2A_1(x)A_2(x), \\ A_2(2x) &= (A_2(x))^2 - (A_1(x))^2 = 2(A_2(x))^2 - I. \end{aligned}$$

Although the matrices found via this method need not be nonsingular, in general, the matrix $A_2(x)$ is, as $A_2(x) = I - \frac{A^2x^2}{2}$ and $\frac{A^2x^2}{2}$ has trace 0. We can also invert $A_2(x)$ and obtain an analogue of the secant-tangent trigonometric square identity:

$$(A_2^{-1}(x))^2 = I + (A_1(x)A_2^{-1}(x))^2.$$

As a second example of our approach, let us solve the problem stated at the beginning of this Capsule. We seek a 4×4, nonconstant matrix function $N(x)$ satisfying the functional equation

$$N(2x) - (N(x))^8 = 0.$$

As before, we consider the associated functional equation in nonconstant $\mathcal{C}^\infty(\mathbb{R})$ functions,

$$g(2x) - (g(x))^8 = 0.$$

Now, this condition immediately implies that either $g(0) = 0$ or $(g(0))^7 = 1$. In the former case, g would vanish to some order $k > 0$ at the origin, and we would then have $k = 8k$ from the functional equation—impossible. Thus, $g(0)$ is a seventh root of unity, and we may require $g(0) = 1$ by considering the function $g/g(0)$ if necessary.

Then, we write $g(x) = 1 + g_nx^n + \cdots$ for some minimal $n > 0$ and $g_n \neq 0$. By the functional equation, we must have

$$1 + 2^ng_nx^n + \cdots = g(2x) = (g(x))^8 = (1 + g_nx^n + \cdots)^8 = 1 + 8g_nx^n + \cdots .$$

Equating coefficients on both sides then gives that $n = 3$; we may also assume that $g_3 = 1$ by replacing x by $x/\sqrt[3]{g_n}$. If we now write $g(x) = 1 + x^3 + g_mx^m + \cdots$, for $m > 3$ minimal and $g_m \neq 0$, we obtain from a further application of the functional equation that $m = 6$. Thus, for some $g_6 \neq 0$,

$$g(x) = 1 + x^3 + g_6x^6 + \cdots .$$

Since the matrix A chosen above has index of nilpotence $4 < 6$, we have found sufficient information to compute the matrix $N(x) := g(Ax)$. Indeed, we have

$$N(x) = g(Ax) = I + A^3x^3 + g_6A^6x^6 + \cdots = I + A^3x^3$$
$$= \begin{pmatrix} 1 & 0 & 864x^3 & 864x^3 \\ 0 & 1 & -864x^3 & -864x^3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is immediate from this construction that $N(x)$ satisfies the desired functional equation.

Unwinding our technique, we obtain a general approach to problems which ask for matrices satisfying analytic functional equations. Specifically, one can approach a problem asking for matrices $\{A_i(x)\}_{i=1}^n$ satisfying an analytic equation

$$E(A_1(x), \ldots, A_n(x)) \equiv 0$$

by trying to solve the equation E for real-valued functions $\{f_i(x)\}_{i=1}^n$. If any of these solutions $\{f_i(x)\}_{i=1}^n$ satisfy $f_i \in \mathcal{C}^\infty(\mathbb{R})$ for all i, then it is possible to find matrix solutions $\{A_i(x)\}_{i=1}^n$ of any dimension n by applying our method with an $n \times n$ nilpotent matrix.

This approach demonstrates a surprising connection between calculus and linear algebra. It could also serve as an elementary introduction to the theory of functions of matrices.

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