Recent Advances in Generalized Matching Theory

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The Marriage Problem (Gale–Shapley, 1962)

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In a society with a set of men M and a set of women W, how can we arrange marriages so that no agent wishes for a divorce?

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Assumptions

1. Bilateral relationships: only pairs (and possibly singles).
2. Two-sided: men only desire women; women only desire men.
3. Preferences are fully known.

The Deferred Acceptance Algorithm

Step 1

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2. Each woman holds onto her most-preferred acceptable proposal (if any) and rejects all others.

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At termination, no agent wants a divorce!

Stability

Definition

A **matching** μ is a one-to-one correspondence on $M \cup W$ such that

- $\mu(m) \in W \cup \{m\}$ for each $m \in M$,
- $\mu(w) \in M \cup \{w\}$ for each $w \in W$, and
- $\mu^2(i) = i$ for all $i \in M \cup W$.

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- **Individually Rational**: All agents i find their matches $\mu(i)$ acceptable.
- **Unblocked**: There do not exist m, w such that both

$$m \succ_w \mu(w) \quad \text{and} \quad w \succ_m \mu(m).$$

Existence and Lattice Structure

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- *If all men (weakly) prefer stable match* μ *to stable match* ν*, then all women (weakly) prefer* ν *to* μ*.*
- *The man- and woman-proposing deferred acceptance algorithms respectively find the man- and woman-optimal stable matches.*

Opposition of Interests: A Simple Example

$$\succ_{m_1} : w_1 \succ w_2 \succ \emptyset \qquad \succ_{w_1} : m_2 \succ m_1 \succ \emptyset$$

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- This opposition of interests result also implies that there is no mechanism which is **strategy-proof** for both men and women.

The "Lone Wolf" Theorem (McVitie–Wilson, 1970)

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- Every woman weakly prefers μ; the number of married women under μ is weakly greater than under $\bar{\mu}$.

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But in general these applications require that women take on multiple partners and that relationships take on many forms.

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- A set of contracts $X \subseteq D \times H \times T$, where T is a finite set of terms such as wages, hours, etc.
  - x_D identifies the doctor of contract x;
  - x_H identifies the hospital of contract x.
- An **outcome** is a set of contracts $Y \subseteq X$ such that if $x, z \in Y$ and $x_D = z_D$, then $x = z$.

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$$R^D(Y) \equiv Y - \cup_{d \in D} C^d(Y),$$
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Definition

The preferences of hospital h are **substitutable** if for all $Y \subseteq X$, if $z \notin C^h(Y)$, then $z \notin C^h(\{x\} \cup Y)$ for all $x \neq z$.

Equilibrium

Definition

An outcome A is **stable** if it is

1. **Individually rational**:
   - for all $d \in D$, if $x \in A$ and $x_D = d$, then $x \succ_d \emptyset$,
   - for all $h \in H$, $C^h(A) = A_H$.
2. **Unblocked**: There does not exist a nonempty **blocking set** $Z \subseteq X - A$ and hospital h such that $Z \subseteq C^h(A \cup Z)$ and $Z \subseteq C^D(A \cup Z)$.

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- Stability is a price-theoretic notion:
  - Every contract not taken . . .
  - . . . is available to some agent who does not choose it.

Characterization of Stable Outcomes

- Consider the operator

$$\Phi_H\left(X^D\right) \equiv X - R_D\left(X^D\right)$$
$$\Phi_D\left(X^H\right) \equiv X - R_H\left(X^H\right)$$
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Theorem

Suppose that the preferences of hospitals are substitutable. Then if $\Phi\left(X^D, X^H\right) = \left(X^D, X^H\right)$, the outcome $X^D \cap X^H$ is stable. Conversely, if A is a stable outcome, there exist $X^D, X^H \subseteq X$ such that $\Phi\left(X^D, X^H\right) = \left(X^D, X^H\right)$ and $X^D \cap X^H = A$.

Existence of Stable Allocations

Theorem

Suppose that hospitals' preferences are substitutable. Then there exists a nonempty finite lattice of fixed points $\left(X^D, X^H\right)$ *of* Φ *which correspond to stable outcomes* $A = X^D \cap X^H$.

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- The proof follows from the isotonicity of the operator Φ.
- The lattice result implies opposition of interests.

The Law of Aggregate Demand

Definition

The preferences of $h \in H$ satisfy the **Law of Aggregate Demand (LoAD)** if for all $Y' \subseteq Y \subseteq X$,

$$\left|C^h(Y)\right| \geq \left|C^h(Y')\right|.$$

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- Intuition: When h receives new offers, he hires at least as many doctors as he did before: no doctor can do the work of two.

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Theorem

If all hospitals' preferences are substitutable and satisfy the LoAD, then each doctor and hospital signs the same number of contracts at each stable outcome.

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If all hospitals' preferences are substitutable and satisfy the LoAD, the doctor-optimal stable many-to-one matching mechanism is (group) strategy-proof.

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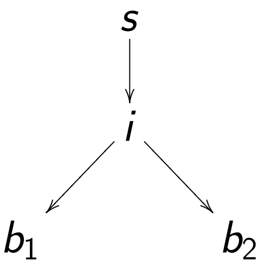
- This has important applications: Sönmez and Switzer (2011), Sönmez (2011) consider the matching of cadets to U.S. Army branches, where preferences are not substitutable, but are *unilaterally substitutable*.
- Open question: What is the necessary and sufficient condition for matching with contracts?

Supply Chain Matching (Ostrovsky, 2008)



- Same-side contracts are *substitutes*.
- Cross-side contracts are *complements*.

$\Rightarrow$ Objects are **fully substitutable**.

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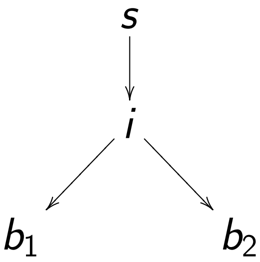

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Stable outcomes exist.

Full Substitutability is Essential (Hatfield–Kominers, 2012)

- Although (full) substitutability is not necessary for many-to-one matching with contracts, it *is* necessary for
  - supply chain matching, and
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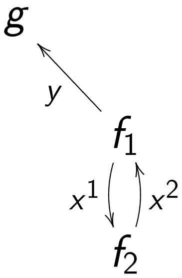
  - supply chain matching, and
  - many-to-many matching with contracts.
- This poses a problem for *couples matching*.
- But new large-market results may provide a partial solution: Kojima–Pathak–Roth (2011); Ashlagi–Braverman–Hassidim (2011); Azevedo–Weyl–White (2012); Azevedo–Hatfield (in preparation).

Cyclic Contract Sets



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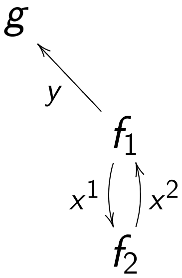

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Acyclicity is necessary for stability.

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In many-to-one (or -many) matching with contracts, if all preferences are substitutable and satisfy the LoAD, then each doctor and hospital signs the same number of contracts at each stable outcome.

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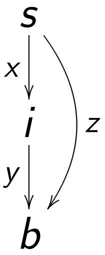
- What happens in supply chains?



$$P^s : \{x\} \succ \{z\} \succ \varnothing$$

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Theorem (supply chain)

Suppose that X is acyclic and that all preferences are fully substitutable and satisfy LoAD (and LoAS). Then, for each agent $f \in F$, the difference between the number of contracts the f buys and the number of contracts f sells is invariant across stable outcomes.

The Model (Koopmans–Beckmann, 1957; Gale, 1960; Shapley–Shubik, 1972)

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Stable assignment $(\tilde{a}_{m,w})$ solves the integer program

$$\max \sum_m \sum_w a_{m,w}\zeta_{m,w} \qquad \left| \begin{array}{ll} 0 \le \sum_w a_{m,w} \le 1 & \forall m \\ 0 \le \sum_m a_{m,w} \le 1 & \forall w \end{array} \right.$$

"Efficient Mating"

- $z_{m,w} \equiv \zeta_{m,w} - \zeta_{m,\emptyset} - \zeta_{\emptyset,w}$ ~ marital surplus

$$\max \sum_{m} \sum_{w} a_{m,w} \zeta_{m,w} = \max \left(\sum_{m} \sum_{w} a_{m,w} z_{m,w} + \sum_{m} \zeta_{m,\emptyset} + \sum_{w} \zeta_{\emptyset,w} \right)$$

Theorem

Stable assignment maximizes aggregate marriage output.

Note

Even with $a_{m,w} \in [0,1]$*, the optimum is always an integer solution.*

Other Notes

- Dual problem shows us "shadow prices" which describe the social cost of removing an agent from the pool of singles.
- If $\zeta_{m,w} = h(x_m, y_w)$, then complementarity (substitution) in traits leads to positive (negative) assortative mating. (Becker, 1973)
- Matches stable in the presence of transfers need not be stable if transfers are not allowed, and vice versa. (Jaffe–Kominers, tomorrow)

Generalization to Networks

Main Results

In **arbitrary** *trading networks with*

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2. *transferable utility, and*
3. *fully substitutable preferences,*

competitive equilibria exist and coincide with stable outcomes.

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- Full substitutability is necessary for these results.
- Correspondence results extend to other solutions concepts.

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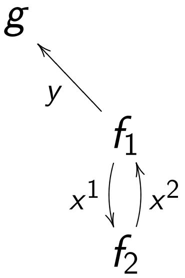

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Related Literature

Matching:

- *Kelso–Crawford (1982)*: Many-to-one (with transfers); (GS)
- *Ostrovsky (2008)*: Supply chain networks; (SSS) and (CSC)
- *Hatfield–Kominers (2012)*: Trading networks (sans transfers)

Exchange economies with indivisibilities:

- *Koopmans–Beckmann (1957)*; *Shapley–Shubik (1972)*
- *Gul–Stachetti (1999)*: (GS)
- *Sun–Yang (2006, 2009)*: (GSC)

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- An **arrangement** is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^{|\Omega|}$.

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- An **arrangement** is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^{|\Omega|}$.
- Set of **contracts** $X := \Omega \times \mathbb{R}$
  - each contract $x \in X$ is a pair (ω, p_ω)
  - $\tau(Y) \subseteq \Omega \sim$ set of trades in contract set $Y \subseteq X$
- A **(feasible) outcome** is a set of contracts $A \subseteq X$ which uniquely prices each trade in A.

The Setting: Demand

- Each agent i has quasilinear utility over arrangements:

$$U_i([\Psi; p]) = u_i(\Psi_i) + \sum_{\psi \in \Psi_{i\rightarrow}} p_\psi - \sum_{\psi \in \Psi_{\rightarrow i}} p_\psi .$$

- U_i extends naturally to (feasible) outcomes.

The Setting: Demand

- Each agent i has quasilinear utility over arrangements:

$$U_i([\Psi; p]) = u_i(\Psi_i) + \sum_{\psi \in \Psi_{i\rightarrow}} p_\psi - \sum_{\psi \in \Psi_{\rightarrow i}} p_\psi.$$

- U_i extends naturally to (feasible) outcomes.

- For any price vector $p \in \mathbb{R}^{|\Omega|}$, the **demand** of i is

$$D_i(p) = \text{argmax}_{\Psi \subseteq \Omega_i} U_i([\Psi; p]).$$

- For any set of contracts $Y \subseteq X$, the **choice** of i is

$$C_i(Y) = \text{argmax}_{Z \subseteq Y_i} U_i(Z).$$

Assumptions on Preferences

1. $u_i(\Psi) \in \mathbb{R} \cup \{-\infty\}$.

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2. $u_i(\varnothing) \in \mathbb{R}$.
3. **Full substitutability**...

Full Substitutability (I)

Definition

The preferences of agent i are **fully substitutable** (in **choice language**) if

1. same-side contracts are substitutes for i, and
2. cross-side contracts are complements for i.

Full Substitutability (I)

Definition

The preferences of agent i are **fully substitutable** (in **choice language**) if for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$,

1. if $Y_{i\rightarrow} = Z_{i\rightarrow}$, and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, then for $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{\rightarrow i} - Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} - Z^*_{\rightarrow i})$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;
2. if $Y_{\rightarrow i} = Z_{\rightarrow i}$, and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, then for $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i\rightarrow} - Y^*_{i\rightarrow}) \subseteq (Z_{i\rightarrow} - Z^*_{i\rightarrow})$ and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$.

Full Substitutability (II)

Theorem

Choice-language full substitutability

1. *has equivalents in* **demand** *and* **"indicator"** *languages;*
2. *holds if and only if the indirect utility function*

$$V_i(p) := \max_{\Psi \subseteq \Omega_i} U_i([\Psi; p])$$

is submodular $(V_i(p \vee q) + V_i(p \wedge q) \leq V_i(p) + V_i(q))$.

Solution Concepts

Definition

An outcome A is **stable** if it is

1. **Individually rational**: for each $i \in I$, $A_i \in C_i(A)$;
2. **Unblocked**: There is no nonempty, feasible $Z \subseteq X$ such that
    - $Z \cap A = \varnothing$ and
    - for each i, and for each $Y_i \in C_i(Z \cup A)$, we have $Z_i \subseteq Y_i$.

Definition

Arrangement $[\Psi; p]$ is a **competitive equilibrium (CE)** if for each i,

$$\Psi_i \in D_i(p).$$

Existence of Competitive Equilibria

Theorem

If preferences are fully substitutable, then a CE exists.

Proof

1. *Modify*: Transform potentially unbounded u_i to $\hat{u}_i$.
2. *Associate*: Construct a two-sided one-to-many matching market:
$$\begin{cases} i \longrightarrow \text{"firm": valuation } \tilde{u}_i(\Psi) := \hat{u}_i(\Psi_{\rightarrow i} \cup (\Omega - \Psi)_{i\rightarrow}); \\ \omega \longrightarrow \text{"worker": wants high wages;} \\ p \longrightarrow \text{"wage".} \end{cases}$$
3. A CE exists in the associated market (Kelso–Crawford, 1982).
4. CE associated $\longrightarrow$ CE modified $=$ CE original.

Structure of Competitive Equilibria

Theorem (First Welfare Theorem)

Let $[\Psi; p]$ *be a CE. Then* Ψ *is efficient.*

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Suppose agents' preferences are fully substitutable. Then, for any CE $[\Xi; p]$ and efficient set of trades Ψ, $[\Psi; p]$ is a CE.

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Theorem (Lattice Structure)

The set of CE price vectors is a lattice.

The Relationship Between Stability and CE

Theorem

If $[\Psi; p]$ *is a CE, then* $A \equiv \cup_{\psi \in \Psi}\{(\psi, p_\psi)\}$ *is stable.*

- *The reverse implication is not true in general.*

The Relationship Between Stability and CE

Theorem

If $[\Psi; p]$ is a CE, then $A \equiv \cup_{\psi \in \Psi}\{(\psi, p_\psi)\}$ is stable.

- *The reverse implication is not true in general.*

Theorem

Suppose that agents' preferences are fully substitutable and A is stable. Then, there exists a price vector $p \in \mathbb{R}^{|\Omega|}$ such that

1. *$[\tau(A); p]$ is a CE, and*
2. *if $(\omega, \bar{p}_\omega) \in A$, then $p_\omega = \bar{p}_\omega$.*

Full Substitutability is Necessary

Theorem

Suppose that there exist at least four agents and that the set of trades is exhaustive. Then, if the preferences of some agent i are not fully substitutable, there exist "simple" preferences for all agents $j \neq i$ such that no stable outcome exists.

Full Substitutability is Necessary

Theorem

Suppose that there exist at least four agents and that the set of trades is exhaustive. Then, if the preferences of some agent i are not fully substitutable, there exist "simple" preferences for all agents $j \neq i$ such that no stable outcome exists.

Corollary

Under the conditions of the above theorem, there exist "simple" preferences for all agents $j \neq i$ such that no CE exists.

Alternative Solution Concepts

Definition

An outcome A is in the **core** if there is no group deviation Z such that $U_i(Z) > U_i(A)$ for all i associated with Z.

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Definition

A set of contracts Z is a **chain** if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^{\ell+1}) = b(y^{\ell})$ for all $\ell < |Z|$.

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Definition

Outcome A is **stable** if it is individually rational and

- **Unblocked**: There is no nonempty, feasible $Z \subseteq X$ such that
  - $Z \cap A = \varnothing$ and
  - for each i, and for each $Y_i \in C_i(Z \cup A)$, we have $Z_i \subseteq Y_i$.

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An outcome A is in the **core** if there is no group deviation Z such that $U_i(Z) > U_i(A)$ for all i associated with Z.

Definition

A set of contracts Z is a **chain** if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^{\ell+1}) = b(y^\ell)$ for all $\ell < |Z|$.

Definition

Outcome A is **chain stable** if it is individually rational and

- **Unblocked**: There is no nonempty, feasible chain $Z \subseteq X$ s.t.
  - $Z \cap A = \varnothing$ and
  - for each i, and for each $Y_i \in C_i(Z \cup A)$, we have $Z_i \subseteq Y_i$.

Alternative Solution Concepts

Definition

An outcome A is in the **core** if there is no group deviation Z such that $U_i(Z) > U_i(A)$ for all i associated with Z.

Definition

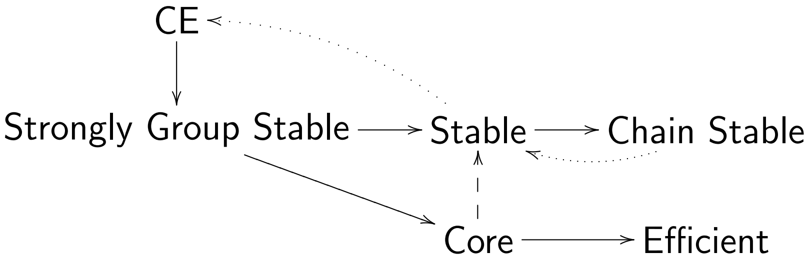
A set of contracts Z is a **chain** if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^{\ell+1}) = b(y^{\ell})$ for all $\ell < |Z|$.

Definition

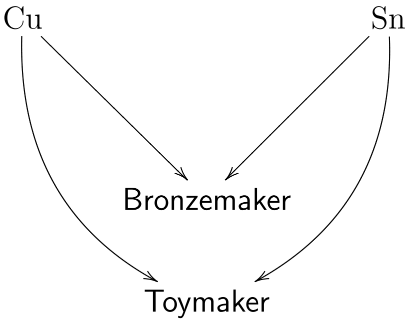
Outcome A is **strongly group stable** if it is individually rational and

- **Unblocked**: There is no nonempty, feasible $Z \subseteq X$ such that
  - $Z \cap A = \varnothing$ and
  - for each i associated with Z, there exists a $Y^i \subseteq Z \cup A$ such that $Z_i \subseteq Y^i$ and $U_i(Y^i) > U_i(A)$.

Relationship Between the Concepts

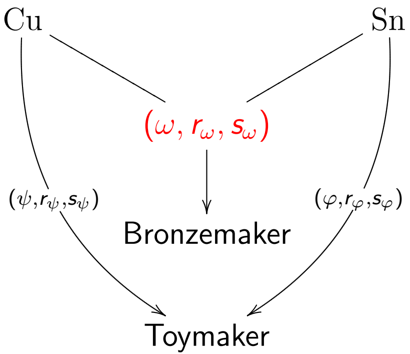


Multilateral Contracts



- Full substitutability is “necessary” in (Discrete, Bilateral) Contract Matching with Transfers.

Multilateral Contracts



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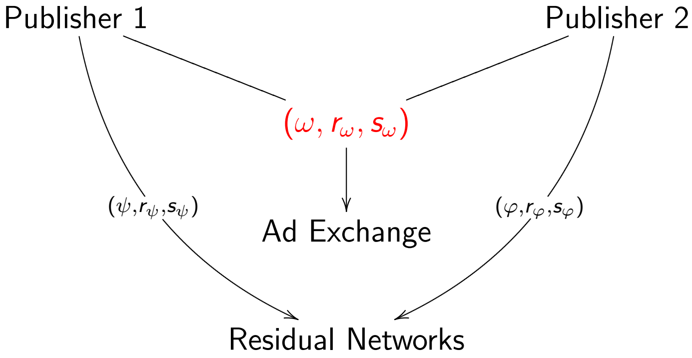

- Full substitutability is "necessary" in (Discrete, Bilateral) Contract Matching with Transfers.

Multilateral Contracts

Main Results

In arbitrary trading networks with

1. **multilateral contracts**,
2. *transferable utility,*
3. **concave** *preferences, and*
4. **continuously divisible contracts**,

competitive equilibria exist and coincide with stable outcomes.

$\Longrightarrow$ Some production complementarities "work" in matching!

Discussion

- Applications of stability in absence of CE?
- Linear programming approach?
- Empirical applications?
- Substitutability vs. concavity?

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- Linear programming approach?
- Empirical applications?
- Substitutability vs. concavity?

\end{Talk}

Demand-Language Full Substitutability

Definition

The preferences of agent i are **fully substitutable** in **demand language** if for all $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$,

1. if $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

$$\Psi_{i\rightarrow} \subseteq \Psi'_{i\rightarrow}, \quad \{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \subseteq \Psi_{\rightarrow i};$$

2. if $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

$$\Psi_{\rightarrow i} \subseteq \Psi'_{\rightarrow i}, \quad \{\omega \in \Psi'_{i\rightarrow} : p_\omega = p'_\omega\} \subseteq \Psi_{i\rightarrow}.$$

Indicator-Language Full Substitutability

$$e^i_\omega(\Psi) = \begin{cases} 1 & \omega \in \Psi_{\rightarrow i} \\ -1 & \omega \in \Psi_{i \rightarrow} \\ 0 & \text{otherwise} \end{cases}$$

Definition

The preferences of agent i are **fully substitutable** in **indicator language** if for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

$$e^i_\omega(\Psi) \leq e^i_\omega(\Psi')$$

for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.