Substitutability in Generalized Matching

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Organization of This Lecture

- (More on) Many-to-One Matching with Contracts
  - Hatfield–Milgrom (2005); Hatfield–Kojima (2008, 2010); Hatfield–K. (2014)
- Many-to-Many Matching with Contracts
  - Hatfield–K. (2012)
- Supply Chain Matching
  - Ostrovsky (2008)
- Fully General Trading Networks (with Transfers)
  - Hatfield–K.–Nichifor–Ostrovsky–Westkamp (2013, ...); Hatfield–K. (forth.)

Focus along the way: Characterizations and Impact of Substitutability

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Focus along the way: Characterizations and Impact of Substitutability

(Please pay attention to notation....)

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- A set of doctors D: each doctor d has a strict preference order P^d over contracts involving him;
- A set of hospitals H: each hospital h has a strict preference order P^h over sets of contracts involving it; and
- A set of contracts $X \subseteq D \times H \times T$, where T is a finite set of terms such as $\{\text{wages}, \text{hours}, \ldots\}$.
  - x_D identifies the doctor of contract x;
  - x_H identifies the hospital of contract x.

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  - x_D identifies the doctor of contract x;
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- An **outcome** is a set of contracts $Y \subseteq X$ such that if $x, z \in Y$ and $x_D = z_D$, then $x = z$.

Substitutability: Review

- $C^d(Y) \equiv \max_{P^d}\{x \in Y : x_D = d\}$.
- $C^h(Y) \equiv \max_{P^h}\{Z \subseteq Y : Z_H = \{h\}\}$.

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Definition

The preferences of hospital h are **substitutable** if for all $x, z \in X$ and $Y \subseteq X$, if $z \notin C^h(Y \cup \{z\})$, then $z \notin C^h(Y \cup \{z, x\})$.

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Definition

Equivalently, the preferences of hospital h are **substitutable** if the rejection function $R^h(Y) \equiv Y \setminus C^h(Y)$ is isotone.

i.e. Gaining a new contract can never make h want to take back a contract it rejected.

Solution Concept

Definition

An outcome A is **stable** if it is

1. **Individually rational**:
   - for all $d \in D$, $C^d(A) = A_d$; and
   - for all $h \in H$, $C^h(A) = A_h$.
2. **Unblocked**: There does not exist a nonempty **blocking set** $Z \subseteq X \setminus A$ and hospital h such that $Z \subseteq C^h(A \cup Z)$ and $Z \subseteq C^D(A \cup Z)$.

Existence of Stable Outcomes (I)

Theorem (Hatfield–Milgrom, 2005)

Suppose that hospitals' preferences are substitutable. Then there exists a nonempty finite lattice of fixed points (X^D, X^H) *of the generalized deferred acceptance operator, corresponding to stable outcomes* $A = X^D \cap X^H$.

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Substitutability is *Not Exactly Necessary*. . . .

- Consider the case of one hospital h with preferences

$$\{x^\alpha, z^\beta\} \succ \{x^\beta\} \succ \{z^\beta\} \succ \{x^\alpha\} \succ \varnothing,$$

  which are not substitutable.

For any choice of doctor preferences, there exists a stable outcome!

Weaker Substitutability Conditions

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The preferences of hospital h are **unilaterally substitutable** if for all $z, x \in X$ and $Y \subseteq X$ for which $z_D \notin Y_D$, if $z \notin C^h(Y \cup \{z\})$, then $z \notin C^h(Y \cup \{z, x\})$.

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Definition

The preferences of hospital h are **weakly substitutable** if for all $z, x \in X$ and $Y \subseteq X$ for which $z_D, x_D \notin Y_D$ and $|Y| = |Y_D|$, if $z \notin C^h(Y \cup \{z\})$, then $z \notin C^h(Y \cup \{z, x\})$.

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Existence of Stable Outcomes (II)

Theorem (Hatfield–Kojima, 2008)

Suppose that there are at least two hospitals. Then, if the preferences of some hospital h are not weakly substitutable, then there exist unit-demand preferences for all other agents such that no stable outcome exists.

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Theorem (Hatfield–Kojima, 2010)

Suppose that hospitals' preferences are unilaterally substitutable. Then the usual results for matching with contracts hold ($\{$existence, lattice structure, rural hospitals' theorem under LoAD, $\ldots\}$).

But wait. . . .

- Consider the case of one hospital h with preferences

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  which are not substitutable.

For any choice of doctor preferences, there exists a stable outcome!

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- Consider the case of one hospital h with preferences

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- h actually wants to hire *two Sherlocks*:

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Maybe we should look at many-to-many matching with contracts. . . ?

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  - The same deferred acceptance operator works!
- Under the LoAD (for all agents), we get a Rural Hospitals Theorem.
- This explains why stable many-to-one matching with contracts outcomes exist when h "wants to hire two Sherlocks:"

$$\{S^r, S^c\} \succ \{S^r, W^c\} \succ \{S^c\} \succ \{W^c\} \succ \{S^r\} \succ \varnothing.$$

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- We have to think carefully about how/whether we want to allow multiple contracts between a given doctor–hospital pair:

$$\{S^r, S^c\} \succ \{S^r, W^c\} \succ \{S^c\} \succ \{W^c\} \succ \{S^r\} \succ \varnothing$$

vs.

$$\{S^{r,c}\} \succ \{S^r, W^c\} \succ \{S^c\} \succ \{W^c\} \succ \{S^r\} \succ \varnothing.$$

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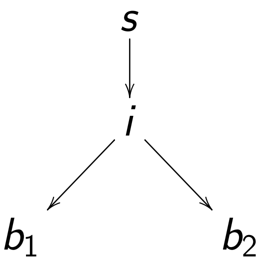
$$\{x^{\$}\} \succ \{x^{w}, x^{\$}\} \succ \varnothing \qquad \{x^{w}\} \succ \{x^{w}, x^{\$}\} \succ \varnothing$$

vs.

$$\{x^{w,\$}\} \succ \varnothing \qquad \{x^{w,\$}\} \succ \varnothing.$$

Supply Chain Matching



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- Cross-side contracts are *complements*.

$\Rightarrow$ Objects are **fully substitutable**.

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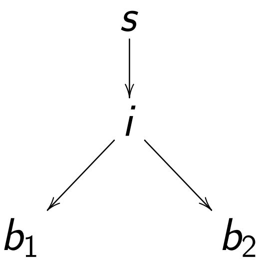

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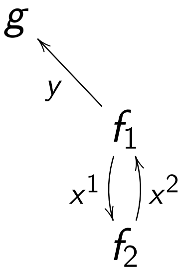
Theorem (Ostrovsky, 2008; Hatfield–K., 2012)

Suppose that all agents' preferences are fully substitutable. Then there exists a nonempty lattice of stable outcomes.

Cyclic Contract Sets



$$P^{f_1} : \{y, x^2\} \succ \{x^1, x^2\} \succ \varnothing$$

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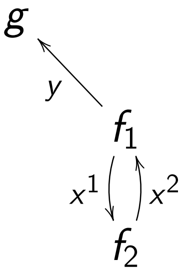

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Theorem

Acyclicity is necessary for stability.

The Rural Hospitals Theorem

Theorem (two-sided)

In many-to-one (or -many) matching with contracts, if all preferences are substitutable and satisfy the LoAD, then each doctor and hospital signs the same number of contracts at each stable outcome.

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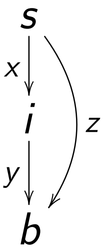
- What happens in supply chains?



$$P^s : \{x\} \succ \{z\} \succ \varnothing$$

$$P^i : \{x, y\} \succ \varnothing$$

$$P^b : \{z\} \succ \{y\} \succ \varnothing$$

The Rural Hospitals Theorem

Theorem (two-sided)

In many-to-one (or -many) matching with contracts, if all preferences are substitutable and satisfy the LoAD, then each doctor and hospital signs the same number of contracts at each stable outcome.

Theorem (supply chain)

Suppose that X is acyclic and that all preferences are fully substitutable and satisfy the LoAD (and LoAS). Then, for each agent $f \in F$, the difference between the number of contracts f buys and the number of contracts f sells is invariant across stable outcomes.

Generalization to Networks

Main Results

In **arbitrary** *trading networks with*

1. *bilateral contracts,*
2. *transferable utility, and*
3. *fully substitutable preferences,*

competitive equilibria exist and coincide with stable outcomes.

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Main Results

In **arbitrary** *trading networks with*

1. *bilateral contracts,*
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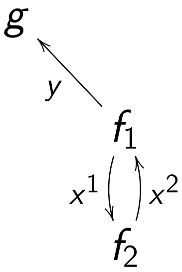
competitive equilibria exist and coincide with stable outcomes.

- Full substitutability is necessary for these results.
- Correspondence results extend to other solutions concepts.

Cyclic Contract Sets



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Theorem

Acyclicity is necessary for stability!

Related Literature

Matching:

- ✓ *Kelso–Crawford (1982)*: Many-to-one (with transfers); (GS)
- ✓ *Ostrovsky (2008)*: Supply chain networks; (SSS) and (CSC)
- ✓ *Hatfield–K. (2012)*: Trading networks (sans transfers)

Exchange economies with indivisibilities:

- *Koopmans–Beckmann (1957)*; *Shapley–Shubik (1972)*
- *Gul–Stachetti (1999)*: (GS)
- *Sun–Yang (2006, 2009)*: (GSC)

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- An **arrangement** is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^{|\Omega|}$.

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- An **arrangement** is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^{|\Omega|}$.
- Set of **contracts** $X := \Omega \times \mathbb{R}$
  - each contract $x \in X$ is a pair (ω, p_ω)
  - $\tau(Y) \subseteq \Omega \sim$ set of trades in contract set $Y \subseteq X$
- A **(feasible) outcome** is a set of contracts $A \subseteq X$ which uniquely prices each trade in A.

The Setting: Demand

- Each agent i has quasilinear utility over arrangements:

$$U_i\left([\Psi; p]\right) = u_i(\Psi_i) + \sum_{\psi \in \Psi_{i\rightarrow}} p_\psi - \sum_{\psi \in \Psi_{\rightarrow i}} p_\psi .$$

- U_i extends naturally to (feasible) outcomes.

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- U_i extends naturally to (feasible) outcomes.

- For any price vector $p \in \mathbb{R}^{|\Omega|}$, the **demand** of i is

$$D_i(p) = \text{argmax}_{\Psi \subseteq \Omega_i} U_i([\Psi;p]).$$

- For any set of contracts $Y \subseteq X$, the **choice** of i is

$$C_i(Y) = \text{argmax}_{Z \subseteq Y_i} U_i(Z).$$

Assumptions on Preferences

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3. **Full substitutability**...

Full Substitutability (I)

Definition

The preferences of agent i are **fully substitutable** (in **choice language**) if

1. same-side contracts are substitutes for i, and
2. cross-side contracts are complements for i.

Full Substitutability (I)

Definition

The preferences of agent i are **fully substitutable** (in **choice language**) if for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$,

1. if $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, then for $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{\to i} \setminus Y^*_{\to i}) \subseteq (Z_{\to i} \setminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;
2. if $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, then for $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i\to} \setminus Y^*_{i\to}) \subseteq (Z_{i\to} \setminus Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.

Full Substitutability (II)

Definition

The preferences of agent i are **fully substitutable** in **demand language** if for all $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$,

1. if $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

$$\Psi_{i\to} \subseteq \Psi'_{i\to}, \quad \{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i};$$

2. if $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$, and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

$$\Psi_{\to i} \subseteq \Psi'_{\to i}, \quad \{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}.$$

Full Substitutability (III)

Definition

The preferences of agent *i* are **fully substitutable** in **"indicator language"** if

- *i* is more willing to "demand" a trade ω (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of trades $\psi \neq \omega$ increase.

Full Substitutability (IV)

Theorem

All three full substitutability notions are equivalent, and hold if and only if the indirect utility function

$$V_i(p) := \max_{\Psi \subseteq \Omega_i} U_i([\Psi; p])$$

is submodular ($V_i(p \vee q) + V_i(p \wedge q) \leq V_i(p) + V_i(q)$).

Solution Concepts

Definition

An outcome A is **stable** if it is

1. **Individually rational**: for each $i \in I$, $A_i \in C_i(A)$;
2. **Unblocked**: There is no nonempty, feasible $Z \subseteq X$ such that
   - $Z \cap A = \varnothing$ and
   - for each i, and for each $Y_i \in C_i(Z \cup A)$, we have $Z_i \subseteq Y_i$.

Definition

Arrangement $[\Psi; p]$ is a **competitive equilibrium (CE)** if for each i,

$$\Psi_i \in D_i(p).$$

Existence of Competitive Equilibria

Theorem

If preferences are fully substitutable, then a CE exists.

Proof

1. *Modify*: Transform potentially unbounded u_i to $\hat{u}_i$.
2. *Associate*: Construct a two-sided one-to-many matching market:

$$\begin{cases} i \to \text{“firm”: valuation } \tilde{u}_i(\Psi) := \hat{u}_i(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}); \\ \omega \to \text{“worker”: wants high wages;} \\ p \to \text{“wage”.} \end{cases}$$

3. A CE exists in the associated market (Kelso–Crawford, 1982).
4. CE associated $\to$ CE modified $=$ CE original.

Structure of Competitive Equilibria

Theorem (First Welfare Theorem)

Let $[\Psi; p]$ *be a CE. Then* Ψ *is efficient.*

Structure of Competitive Equilibria

Theorem (First Welfare Theorem)

Let $[\Psi; p]$ be a CE. Then Ψ is efficient.

Theorem (Second Welfare Theorem)

Suppose agents' preferences are fully substitutable. Then, for any CE $[\Xi; p]$ and efficient set of trades Ψ, $[\Psi; p]$ is a CE.

Structure of Competitive Equilibria

Theorem (First Welfare Theorem)

Let $[\Psi; p]$ be a CE. Then Ψ is efficient.

Theorem (Second Welfare Theorem)

Suppose agents' preferences are fully substitutable. Then, for any CE $[\Xi; p]$ and efficient set of trades Ψ, $[\Psi; p]$ is a CE.

Theorem (Lattice Structure)

The set of CE price vectors is a lattice.

The Relationship Between Stability and CE (I)

Theorem

If $[\Psi; p]$ *is a CE, then* $A \equiv \cup_{\psi \in \Psi}\{(\psi, p_\psi)\}$ *is stable.*

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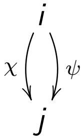

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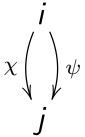

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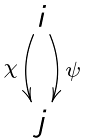

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$\Rightarrow$ $\varnothing$ is a stable outcome, but no CE exists.

The Relationship Between Stability and CE (II)

Theorem

Suppose that agents' preferences are fully substitutable and A is stable. Then, there exists a price vector $p \in \mathbb{R}^{|\Omega|}$ such that

1. $[\tau(A); p]$ *is a CE, and*
2. *if* $(\omega, \bar{p}_\omega) \in A$*, then* $p_\omega = \bar{p}_\omega$*.*

Proof

Full subs. $\Rightarrow$ CE of economy with trades $\Omega \setminus \tau(A)$ and valuations

$$\hat{u}_i(\Psi) = \max_{Y \subseteq A_i} \left[u_i(\Psi \cup \tau(Y)) + \sum_{(\omega,\bar{p}_\omega)\in Y_{i\rightarrow}} \bar{p}_\omega - \sum_{(\omega,\bar{p}_\omega)\in Y_{\rightarrow i}} \bar{p}_\omega \right].$$

Find CE of the form $[\varnothing; q_{\Omega\setminus\tau(A)}]$; then take $p = (\bar{p}_{\tau(A)}, q_{\Omega\setminus\tau(A)})$.

Full Substitutability is Necessary

Theorem

Suppose that there exist at least four agents and that the set of trades is exhaustive. Then, if the preferences of some agent i are not fully substitutable, there exist "simple" preferences for all agents $j \neq i$ such that no stable outcome exists.

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Corollary

Under the conditions of the above theorem, there exist "simple" preferences for all agents $j \neq i$ such that no CE exists.

Alternative Solution Concepts

Definition

An outcome A is in the **core** if there is no group deviation Z such that $U_i(Z) > U_i(A)$ for all i associated with Z.

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Outcome A is **stable** if it is individually rational and

- **Unblocked**: There is no nonempty, feasible $Z \subseteq X$ such that
  - $Z \cap A = \varnothing$ and
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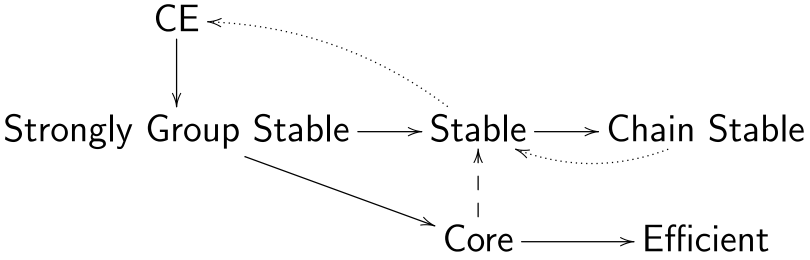
Definition

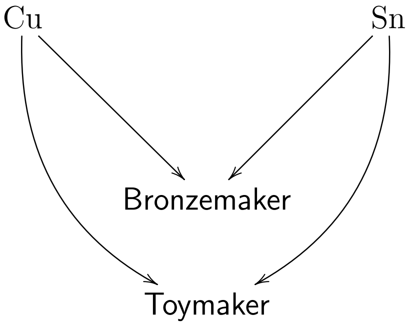
Outcome A is **strongly group stable** if it is individually rational and

- **Unblocked**: There is no nonempty, feasible $Z \subseteq X$ such that
  - $Z \cap A = \varnothing$ and
  - for each i associated with Z, there exists a $Y^i \subseteq Z \cup A$ such that $Z_i \subseteq Y^i$ and $U_i(Y^i) > U_i(A)$.

Relationship Between the Concepts



Multilateral Contracts



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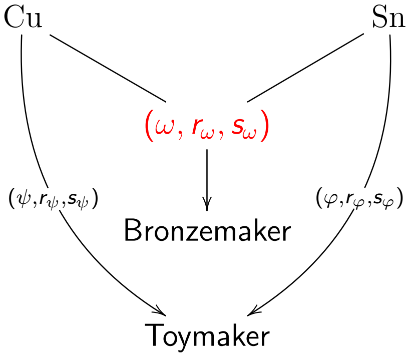

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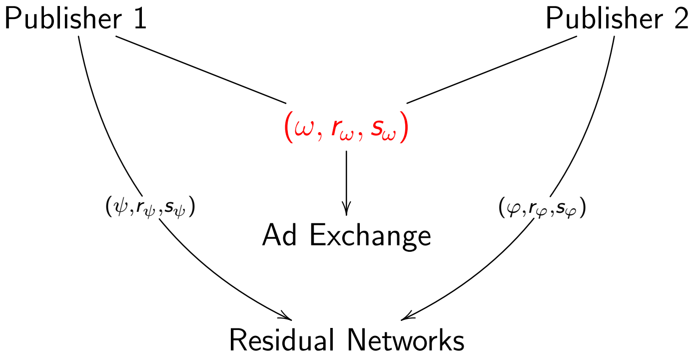

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Multilateral Contracts

Main Results

In arbitrary trading networks with

1. **multilateral contracts**,
2. *transferable utility,*
3. **concave** *preferences, and*
4. **continuously divisible contracts**,

competitive equilibria exist and coincide with stable outcomes.

$\Longrightarrow$ Some production complementarities “work” in matching!

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- Generalized matching $\rightsquigarrow$ design of affirmative action programs (K.–Sönmez, 2013; Dur–K.–Pathak–Sönmez, 2013).
- Stable outcomes give sharp predictions for quality compeition in the presence of price restrictions (Hatfield–Plott–Tanaka, 2013).

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- Applications of stability in absence of CE?
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- Empirical applications?
- Substitutability vs. concavity?

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\end{Lecture}