Configurations of Extremal Type II Lattices and Codes

Scott Duke Kominers

Department of Economics, Harvard University, and Harvard Business School

AMS-MAA-SIAM Session on Research in Mathematics by Undergraduates

Joint Mathematics Meetings

January 15, 2010

Key Concepts

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lattice

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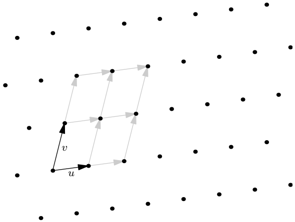

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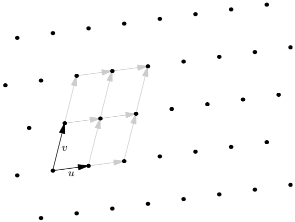

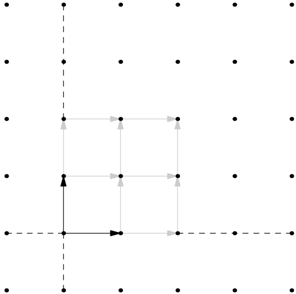

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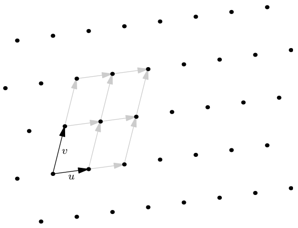

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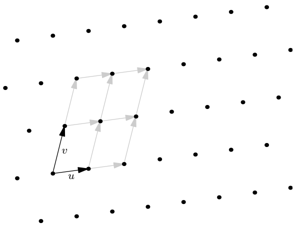

- "integer vector space"

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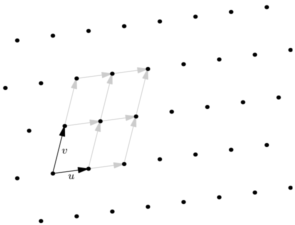

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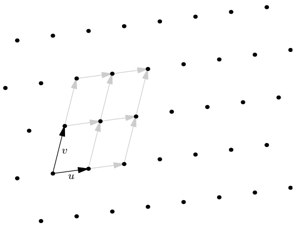

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Key Concepts

lattice

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- lattice ∼ "integer vector space"
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Key Concepts

unimodular lattice

- unimodular $\sim$ self-dual
    - basis matrix has determinant 1
- lattice $\sim$ "integer vector space"
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Key Concepts

even unimodular lattice

- even $\sim$ all vectors have even norm
  - $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$
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Theorem

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- What:

  $$\Theta_L(\tau) = \sum_{x \in L} e^{i\pi\tau\langle x,x\rangle} = \sum_{k=1}^{\infty} a_k e^{i\pi\tau(k)}$$

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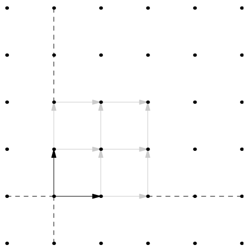

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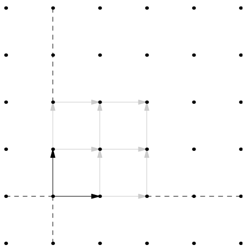

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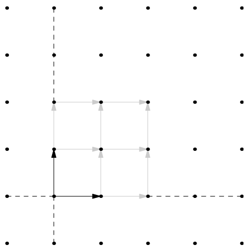

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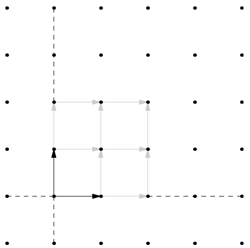

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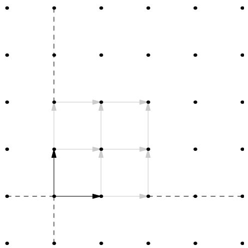

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  - We can therefore study the function Θ_L even if we cannot write down a basis for L.

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- What we do:
  - We study *weighted theta functions* $\sum_{x \in L} P(x) e^{i\pi\tau\langle x,x\rangle}$ which encode norms and distributions of lattice vectors.

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- What we do:
  - We study *weighted theta functions* $\sum_{x \in L} P(x) e^{i\pi\tau\langle x,x\rangle}$ which encode norms and distributions of lattice vectors.
  - We obtain a "system of equations in vector distributions" which proves our configuration results.

Lattice Configuration Results

Theorem Template

If L is Type II and extremal of rank n, then the minimal-norm vectors of L generate L.

- Folklore: $n \in \{8, 24\}$
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If L is Type II and extremal of rank n with minimal norm $m(L)$, then L is generated by its vectors of norms $m(L)$ and $(m(L) + 2)$.

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- Abel–K. (2008): $n \in \{40, 80, 120\}$ (unified method)
- Elkies–K. (2010): Norm-$(m(L)+2)$ suffices for $n \in \{40, 80\}$

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Natural Question

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Natural Question

What about codes?

Key Concepts

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Key Concepts

$$\text{extremal } \underbrace{\text{even unimodular}}_{\text{Type II}} \text{ lattice}$$

- lattice of rank $n \sim$ "integer vector space" of rank n
- code of length $n \sim$ linear subspace of $\mathbb{F}_2^n$

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- unimodular ∼ self-dual
- self-dual ∼ self-dual

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- even $\sim$ all vectors have even norm
- doubly-even $\sim$ 4 divides all codewords' weights

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- extremal $\sim$ shortest vector is as long as possible
- extremal $\sim$ smallest codeword is as large as possible

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Key Concepts

extremal even unimodular lattice

(even unimodular = Type II)

↑ Construction A

extremal doubly-even self-dual code

(doubly-even self-dual = Type II)

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  encodes the distributions of L's vectors."

- `Weight Enumerator Slogan:`

  "The *harmonic weight enumerator* $W_{C,Q}(x, y)$ of C
  is a classifiable polynomial which
  encodes the distributions of C's codewords."

Code Configuration Results

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Theorem Template

If C is Type II and extremal of length n, then the minimal-weight codewords of C generate C.

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- Likely: Analog of slightly weaker result for $n \in \{40, 80, 120\}$

Conclusion

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Lattices

Codes

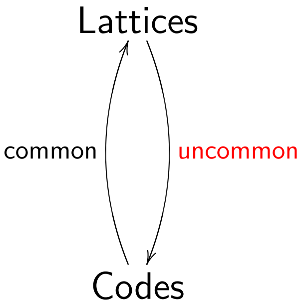
Conclusion

Lattices

common

Codes

Conclusion



Acknowledgments

Acknowledgments

- Prof. Noam D. Elkies

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QED

Questions?

http://www.scottkom.com/

Extra Slides

Example Code

The codewords of the *extended Hamming code* e_8 are given by the columns of the matrix

$$\begin{pmatrix} 0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1\\ 0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1\\ 0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&1&1&1&1&0&0&1&1&0&0&0&0&1&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix}.$$

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- $2^{25}3^9 5^4 7^2 t(168t^4 - 2800t^3 + 17745t^2 - 50635t + 54834) = 0 \Rightarrow t = 0 \Rightarrow\Leftarrow$.