Frontiers of Matching Theory

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The Marriage Problem (Gale–Shapley, 1962)

Question

In a society with

$$1 \text{ man and } 0 \text{ women,}$$

how can we arrange marriages so that there are no divorces?

m

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In a society with

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In a society with

$$3 \text{ men and } 1 \text{ woman,}$$

how can we arrange marriages so that there are no divorces?

m_1 w

m_2

m_3

The Marriage Problem (Gale–Shapley, 1962)

Question

In a society with

$$M \text{ men and } 1 \text{ woman,}$$

how can we arrange marriages so that there are no divorces?

m_1 w

$\vdots$

m_M

The Marriage Problem (Gale–Shapley, 1962)

Question

In a society with

$$M \text{ men and } W \text{ women,}$$

how can we arrange marriages so that there are no divorces?

$$\begin{array}{cc} m_1 & w_1 \\ \vdots & \vdots \\ m_M & w_W \end{array}$$

The Deferred Acceptance Algorithm

Step 1

1. Each man "proposes" to his first-choice woman.
2. Each woman holds onto her most-preferred acceptable proposal (if any) and rejects all others.

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Step $t \geq 2$

1. Each rejected man "proposes" to his next-highest choice woman.
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At termination, no agent wants a divorce!

Stability

Definition

A marriage matching is **stable** if no agent wants a divorce.

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Theorem (Gale–Shapley, 1962)

A stable marriage matching exists.

Lattice Structure: Opposition of Interests

Theorem (Conway, 1976)

- *Given two stable matchings* μ, ν*, there is a stable match* $\mu \vee \nu$ *(*$\mu \wedge \nu$*) which every man likes weakly more (less) than* μ *and* ν*.*

Lattice Structure: Opposition of Interests

Theorem (Conway, 1976)

- *Given two stable matchings μ, ν, there is a stable match $\mu \vee \nu$ ($\mu \wedge \nu$) which every man likes weakly more (less) than μ and ν.*
- *If all men (weakly) prefer stable match μ to stable match ν, then all women (weakly) prefer ν to μ.*

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- *If all men (weakly) prefer stable match μ to stable match ν, then all women (weakly) prefer ν to μ.*
- *The man- and woman-proposing deferred acceptance algorithms respectively find the man- and woman-optimal stable matches.*

Opposition of Interests: A Simple Example

$\succ_{m_1}: w_1 \succ w_2 \succ \emptyset$

$\succ_{m_2}: w_2 \succ w_1 \succ \emptyset$

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woman-optimal stable match

The "Lone Wolf" Theorem (Roth, 1984)

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The set of matched men (women) is invariant across stable matches.

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$$\bar{\mu}_M \qquad \bar{\mu}_W$$

$$\mu_M \qquad \mu_W$$

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Generalizations

- 1962: Many-to-one Matching ("College Admissions")
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Generalizations

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  - Substitutable preferences sufficient for stability
  - "Rural Hospitals" Theorem
- 1985$\pm\varepsilon$: Many-to-many Matching ("Consultants and Firms")
  - Multiple notions of stability
- 2005: Matching with Contracts ("Doctors and Hospitals")
  - $\{\text{Wage}, \text{schedule}, \ldots\}$ negotiations embed into matching

Matching with Contracts (Hatfield–Milgrom, 2005)

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Assumptions

- Hospitals have strict preferences over sets of contracts.
- Doctors have strict preferences and "unit demand."

(Many-to-one) Matching with Contracts (Hatfield–Milgrom)

$$X \subseteq D \times H \times T$$
$$x = (\text{doctor}, \text{hospital}, \text{terms})$$

Special Cases

- Men–Women ($X = M \times W \times \{1\}$; all have unit demand)
- Colleges–Students ($X = S \times C \times \{1\}$)

Substitutability

Definition

The preferences of an agent $f \in D \cup H$ are **substitutable** if there do not exist $x, z \in X$ and $Y \subseteq X$ such that

$$z \notin C^f(Y \cup \{z\}) \quad \text{but} \quad z \in C^f(Y \cup \{x, z\}).$$

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Intuition

Receiving new offers makes f (weakly) less interested in old offers.

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Equivalent Definition

The **rejection function** $R^f(X') = X' - C^f(X')$ is monotone.

Substitutability $\Rightarrow$ Stability

Theorem

Suppose that all preferences are substitutable. Then, the set of stable allocations is a nonempty lattice.

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Proof by "Generalized Deferred Acceptance"

$$\Phi(Y) = X - R^H(X - R^D(Y))$$

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- Correspondence between fixed points Y of Φ and stable allocations $A = C^D(Y)$.

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- If R^H and R^D are monotone, then Φ is monotone.

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Proof by "Generalized Deferred Acceptance"

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- Correspondence between fixed points Y of Φ and stable allocations $A = C^D(Y)$.
- If R^H and R^D are monotone, then Φ is monotone.
- Tarski's Fixed Point Theorem $\implies$ a lattice of fixed points of Φ.

How deep is the rabbit hole?

Question

What is "needed" in order for matching theory to work?

Matching in Networks (Hatfield–K., 2010)

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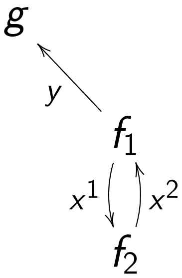
Cyclic Contract Sets



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$$P_g : \{y\} \succ \varnothing$$

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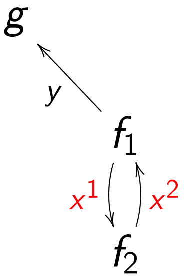

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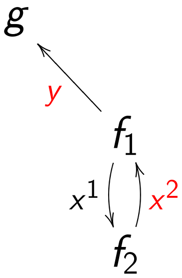

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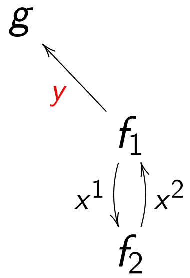

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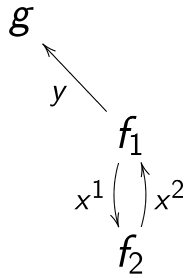

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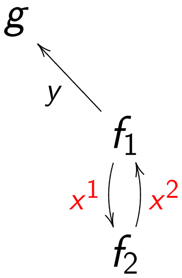

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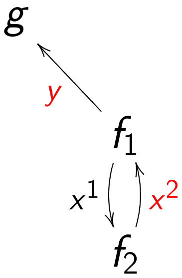

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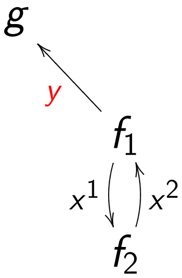

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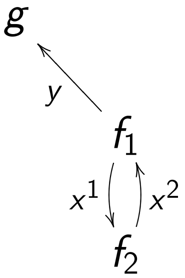

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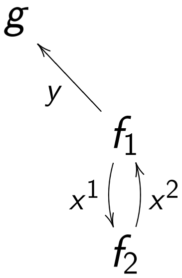

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Theorem

Acyclicity is necessary for stability!

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Assumptions

- Agents have strict preferences over sets of contracts.
- The contract graph is acyclic ($\iff$ supply chain structure).

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Special Cases

- Doctors–Hospitals ($X \subseteq D \times H \times T$)
- Supply chain Matching

Stability

Definition

An allocation of contracts A is **stable** if no set of agents (strictly) prefers to match among themselves than to accept the terms of A.

That is, A is **stable** if it is

1. **Rational**
2. **Unblocked**

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Definition

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Formally: A is **stable** if it is

1. **Rational**: For all $f \in F$, $C^f(A) = A|_f$.
2. **Unblocked**: There does not exist a nonempty **blocking set** $Z \subseteq X$ such that $Z \cap A = \varnothing$ and $Z|_f \subseteq C^f(A \cup Z)$ (for all f).

Substitutability

Definition

The preferences of an agent f are **fully substitutable** if receiving more buyer (seller) contracts makes f

- weakly less interested in his available buyer (seller) contracts and
- weakly more interested in his available seller (buyer) contracts.

Intuition

- same-side contracts are substitutes
- cross-side contracts are complements

Full Substitutability $\Longleftrightarrow$ Guaranteed Stability

Theorem (Sufficiency)

If X is acyclic and all preferences are fully substitutable, then there exists a lattice of stable allocations.

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Theorem (Necessity)

Both conditions in the above theorem are necessary for the result.

Extensions

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  - New conditions sufficient for many-to-one stability

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- Design of **contract language**
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- **Completion** of many-to-one preferences
    - New conditions sufficient for many-to-one stability
- Matching with **money**
    - Pigouvian taxes restore stability for cyclic X

The Law of Aggregate Demand

Definition

Preferences of f satisfy the **Law of Aggregate Demand (LoAD)** if, whenever f receives new offers as a buyer, he takes on at least as many new buyer contracts he does seller contracts.

Intuition

- When f buys a new good, he will sell at most one more good than he was previously selling.
- **Law of Aggregate Supply (LoAS)** is analogous.

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Formally: for all $Y, Y', Z \subseteq X$ such that $Y' \subseteq Y$,

$$\left|C_B^f(Y|Z)\right| - \left|C_B^f(Y'|Z)\right| \geq \left|C_S^f(Z|Y)\right| - \left|C_S^f(Z|Y')\right|.$$

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"Matching in Networks with Bilateral Contracts" (Hatfield–K.)

Bundling of Contract Terms

1. Work and wages contracted simultaneously:
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"Contract Design and Stability in Matching Markets" (Hatfield–K.)

Completion of Preferences

- Consider the case of one hospital h with preferences

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which are not substitutable.

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- This hospital h actually has preferences

$$\succ_h\colon \{x^\alpha, x^\beta\} \succ \{x^\alpha, z^\beta\} \succ \{x^\beta\} \succ \{x^\alpha\} \succ \{z^\beta\},$$

  which ARE substitutable.

Completion of Preferences

- Consider the case of one hospital h with preferences

$$\succ_h\colon \left\{x^\alpha, z^\beta\right\} \succ \left\{x^\beta\right\} \succ \left\{x^\alpha\right\} \succ \left\{z^\beta\right\},$$

  which are not substitutable.

- This hospital h actually has preferences

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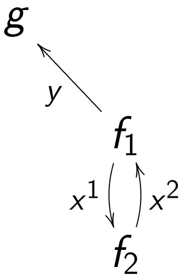
  which ARE substitutable.

"Contract Design and Stability in Matching Markets" (Hatfield–K.)

Cyclic Contract Sets



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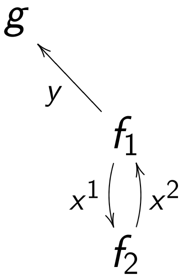
$$P_g : \{y\} \succ \varnothing$$

Theorem

Acyclicity is necessary for stability!

Cyclic Contract Sets



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Acyclicity or transferable utility is necessary for stability!

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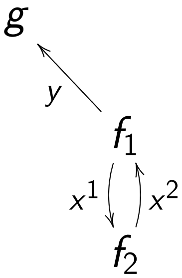

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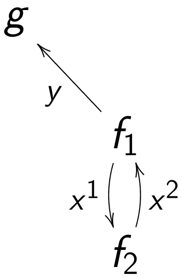

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"Stability and CE in Trading Networks" (Hatfield–K.–Nichifor–Ostrovsky–Westkamp)

Conclusion

Acyclicity and substitutability are necessary and sufficient for (classical) matching theory...

...and at the outer frontiers, surprising structure arises.

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QED

Extra Slides

Related Literature

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- Kara–Sönmez (1996, 1997), Gul–Stachetti (1999), Haake–Klaus (2008a,b), Hatfield–Kojima (2009), Jaume et al. (2009), ...

When *are* preferences substitutable?

- Subdividing reveals Substitutability

$$\succ_h\colon \left\{x^{\alpha}, z^{\beta}\right\} \succ \left\{x^{\alpha,\beta}\right\} \succ \left\{x^{\alpha}\right\} \succ \left\{z^{\beta}\right\} \succ \left\{x^{\beta}\right\}$$
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- Subdividing thwarts Substitutability

$$\succ_d\colon \{x^{40}\} \succ \varnothing$$
$$\succ'_d\colon \{x^{20}, x^{20'}\} \succ \varnothing$$

Substitutability $\Rightarrow$ Stability

Proof by "Generalized Deferred Acceptance"

$$\Phi(Y) = X - R^H(X - R^D(Y))$$

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Substitutability ⇒ Stability

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Full Substitutability $\Rightarrow$ Guaranteed Stability

Proof by "Generalized Deferred Acceptance"

$$\Phi_S\left(X^B, X^S\right) := X - R_B\left(X^B | X^S\right)$$
$$\Phi_B\left(X^B, X^S\right) := X - R_S\left(X^S | X^B\right)$$
$$\Phi\left(X^B, X^S\right) = \left(\Phi_B\left(X^B, X^S\right), \Phi_S\left(X^B, X^S\right)\right)$$

- If X is acyclic, preferences are fully substitutable, and $\Phi\left(X^B, X^S\right) = \left(X^B, X^S\right)$, then $X^B \cap X^S$ stable.
- If X is acyclic, preferences are fully substitutable, and A is stable, then there exist $X^B, X^S \subseteq X$ such that $\Phi\left(X^B, X^S\right) = \left(X^B, X^S\right)$ with $X^B \cap X^S = A$.
- If preferences are fully substitutable, then Φ is isotone.

Chain Stability

Definition

A set of contracts $\{x^1, \ldots, x^N\}$ is a **chain** if

1. $x_B^n = x_S^{n+1}$ for all $n = 1, \ldots, N-1$.
2. $x_S^n = x_S^m$ implies that $n = m$.
3. $x_B^N \neq x_S^1$.

Definition (Ostrovsky, 2008)

An allocation A is **chain stable** if it is individually rational and there is no chain that is a blocking set.

Chain Stability

Theorem

Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then an allocation is stable if and only if it is chain stable.

Corollary

Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then, the set of chain stable allocations is a nonempty lattice.

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But chain stability...

- ...is unappealing when X is cyclic.
  $F = \{f, g\}$; $x_S = y_B = f$; $x_B = y_S = g$;

$$P_f : \{x, y\} \succ \varnothing, \quad P_g : \{x, y\} \succ \varnothing.$$

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- ...is strictly weaker than stability when preferences are not fully substitutable.
- ...does not correspond to standard many-to-many stability.

The Laws of Aggregate Demand and Supply

Definition

The preferences of $f \in F$ satisfy the **Law of Aggregate Demand (LoAD)** if for all $Y, Y', Z \subseteq X$ such that $Y' \subseteq Y$

$$\left|C_B^f(Y|Z)\right| - \left|C_B^f(Y'|Z)\right| \geq \left|C_S^f(Z|Y)\right| - \left|C_S^f(Z|Y')\right|.$$

Definition

The preferences of $f \in F$ satisfy the **Law of Aggregate Supply (LoAS)** if for all $Y, Z, Z' \subseteq X$ such that $Z' \subseteq Z$

$$\left|C_S^f(Z|Y)\right| - \left|C_S^f(Z'|Y)\right| \geq \left|C_B^f(Y|Z)\right| - \left|C_B^f(Y|Z')\right|.$$

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